What are the Most Important Copulas in Finance?

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1 Agenda

- Properties of 2-Copulas
- The most important copulas
- Extreme and singular copulas
- Dependency Bounds
- Application to VaR aggregation
- Application to two assets option pricing

2 Properties of 2-Copulas

Definition 1 (Schweizer and Sklar [1974]) *A two-dimensional copula (or 2-copula) is a function* $\mathbf{C}$ *with the following properties:*

1. $\operatorname{Dom}\mathbf{C} = [0,1] \times [0,1]$;
2. $\mathbf{C}(0,u) = \mathbf{C}(u,0) = 0$ *and* $\mathbf{C}(u,1) = \mathbf{C}(1,u) = u$ *for all* u *in* $[0,1]$;
3. $\mathbf{C}$ *is 2-increasing:*

$$\mathbf{C}(v_1,v_2) - \mathbf{C}(v_1,u_2) - \mathbf{C}(u_1,v_2) + \mathbf{C}(u_1,u_2) \geq 0$$

whenever $(u_1,u_2) \in [0,1]^2$, $(v_1,v_2) \in [0,1]^2$ *such* $0 \leq u_1 \leq v_1 \leq 1$ *and* $0 \leq u_2 \leq v_2 \leq 1$.

$\Rightarrow$ 2-Copulas are also doubly stochastic measures on the unit square.

2.1 Sklar's canonical representation

Theorem 1 *Let* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be 2 univariate distributions. It comes that* $\mathbf{C}\left(\mathbf{F}_1\left(x_1\right), \mathbf{F}_2\left(x_2\right)\right)$ *defines a bivariate probability distribution with margins* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *(because the integral transforms are uniform distributions).*

Theorem 2 *Let* $\mathbf{F}$ *be a 2-dimensional distribution function with margins* $\mathbf{F}_1$ *and* $\mathbf{F}_2$*. Then* $\mathbf{F}$ *has a copula representation:*

$$\mathbf{F}\left(x_1, x_2\right) = \mathbf{C}\left(\mathbf{F}_1\left(x_1\right), \mathbf{F}_2\left(x_2\right)\right)$$

The copula $\mathbf{C}$ *is unique if the margins are continuous. Otherwise, only the subcopula is uniquely determined on* $\operatorname{Ran}\mathbf{F}_1 \times \operatorname{Ran}\mathbf{F}_2$*.*

2.2 Exhaustive statistics of the dependence

The copula function of **random variables** (X_1, X_2) is **invariant** under strictly increasing transformations ($\partial_x h_n(x) > 0$):

$$\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}\langle h_1(X_1), h_2(X_2)\rangle$$

Here are some examples :

$$\begin{aligned}\mathbf{C}\langle X_1, X_2\rangle &= \mathbf{C}\langle \ln X_1, X_2\rangle \\ &= \mathbf{C}\langle \ln X_1, \exp X_2\rangle \\ &= \mathbf{C}\left\langle (X_1 - K_1)^+, (X_2 - K_2)^+\right\rangle\end{aligned}$$

> ... *the copula is invariant while the margins may be changed at will, it follows that is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strickly increasing transformations* (Schweizer and Wolff [1981]).

⇒ **Copula = dependence function of random variables.**

2.3 Topological properties of $\mathcal{C}$

Theorem 3 (Deheuvels [1979]) *The set $\mathcal{C}$ of 2-copulas is compact with any of the following topologies, equivalent on $\mathcal{C}$: punctual convergence, uniform convergence on $[0,1]^2$, weak convergence of the associated probability measure.*

Let $\mathcal{E}x\,(\mathcal{C})$ be the set of the extreme points of $\mathcal{C}$.

$\Rightarrow$ **Choquet's representation of $\mathcal{C}$ similar to the Birkhoff's theorem :**

$$\mathcal{C} \text{ is the convex hull of } \mathcal{E}x\,(\mathcal{C})$$

Problem : What are the equivalent of permutation matrices ?

3 The most important copulas

Correlation = an important tool in Finance. For example, it is sometimes called the diversification coefficient in portfolio analysis.

What are the most important correlations ?

$\rho = -1 \Rightarrow$ "the random variables are completely negatively correlated".

$\rho = 0 \Rightarrow$ "the random variables are uncorrelated".

$\rho = 1 \Rightarrow$ "the random variables are completely correlated".

Problem: Financial people use $\Leftrightarrow$ in the place of $\Rightarrow$.

3.1 $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$

The lower and upper Fréchet bounds $\mathbf{C}^{-}$ and $\mathbf{C}^{+}$ are

$$\begin{aligned}\mathbf{C}^{-}(u_1,u_2) &= \max(u_1+u_2-1,0) \\ \mathbf{C}^{+}(u_1,u_2) &= \min(u_1,u_2)\end{aligned}$$

The product copula corresponds to

$$\mathbf{C}^{\perp}(u_1,u_2) = u_1 u_2$$

We can show that the following order* holds for any copula $\mathbf{C}$:

$$\mathbf{C}^{-} \prec \mathbf{C} \prec \mathbf{C}^{+}$$

$\Rightarrow$ The minimal and maximal distributions of the Fréchet class $\mathcal{F}(\mathbf{F}_1,\mathbf{F}_2)$ are then $\mathbf{C}^{-}(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$ and $\mathbf{C}^{+}(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$.

*$\prec$ is called the concordance order (for distributions) or the stochastic order (for random variables).

3.1.1 Probabilistic interpretation of the three copulas

Mikusiński, Sherwood and Taylor [1991] give the following interpretation of the three copulas $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$:

- Two random variables X_1 and X_2 are **countermonotonic** — or $\mathbf{C} = \mathbf{C}^{-}$ — if there exists a r.v. X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ with f_1 non-increasing and f_2 non-decreasing;
- Two random variables X_1 and X_2 are **independent** if the dependence structure is the product copula $\mathbf{C}^{\perp}$;
- Two random variables X_1 and X_2 are **comonotonic** — or $\mathbf{C} = \mathbf{C}^{+}$ — if there exists a random variable X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where the functions f_1 and f_2 are non-decreasing;

3.1.2 Pearson correlation and copulas

Let X_1 and X_2 two random variables with distributions $\mathbf{F}_1$ and $\mathbf{F}_2$.

$$\rho(X_1, X_2) = \frac{\mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2]}{\sigma[X_1]\,\sigma[X_2]}$$

ρ is also called the Pearson correlation or the **Linear** correlation.

Using works of Tchen [1980] on superadditive functions, we can show the following results:

- If the copula of (X_1, X_2) is $\mathbf{C}^{\perp}$, $\rho(X_1, X_2) = 0$;
- ρ is increasing with respect to the concordance order

$$\mathbf{C}_1 \succ \mathbf{C}_2 \Rightarrow \rho_1(X_1, X_2) \geq \rho_2(X_1, X_2)$$

- $\rho(X_1, X_2)$ is bounded

$$\rho^{-}(X_1, X_2) \leq \rho(X_1, X_2) \leq \rho^{+}(X_1, X_2)$$

and the bounds are attained for the Fréchet copulas $\mathbf{C}^{-}$ and $\mathbf{C}^{+}$.

$$\rho^{-}(X_1, X_2) = \rho^{+}(X_1, X_2) = \frac{\mathbb{E}[f_1(X) f_2(X)] - \mathbb{E}[f_1(X)] \mathbb{E}[f_2(X)]}{\sigma[f_1(X)] \sigma[f_2(X)]}$$

The solution of the equation $\rho^{-}(X_1, X_2) = -1$ (or $\rho^{+}(X_1, X_2) = 1$) is $f_1(x) = a_1 x + b$ and $f_2(x) = a_2 x + b$ with $a_1 a_2 < 0$ ($a_1 a_2 > 0$ for $\rho^{+} = 1$).

$\Rightarrow$ If X_1 and X_2 are not gaussians, there exists very few solutions. For example, if X_1 and X_2 are two log-normal random variables, $\rho^{-} = -1$ can not be reached and $\rho^{+} = 1$ if and only if $\sigma_1 = \sigma_2$.

$\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ are the most important diversification function (in the sense of the correlation). Moreover, we note that

$$\mathbf{C} \succ \mathbf{C}^{\perp} \Rightarrow \rho(X_1, X_2) \geq 0$$

$$\mathbf{C} \prec \mathbf{C}^{\perp} \Rightarrow \rho(X_1, X_2) \leq 0$$

3.2 The Normal copula

The Normal copula is the dependence function of the **gaussian** random vector (X_1, X_2) with correlation parameter ρ:

$$\mathbf{C}_\rho(u_1, u_2) = \Phi\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho\right)$$

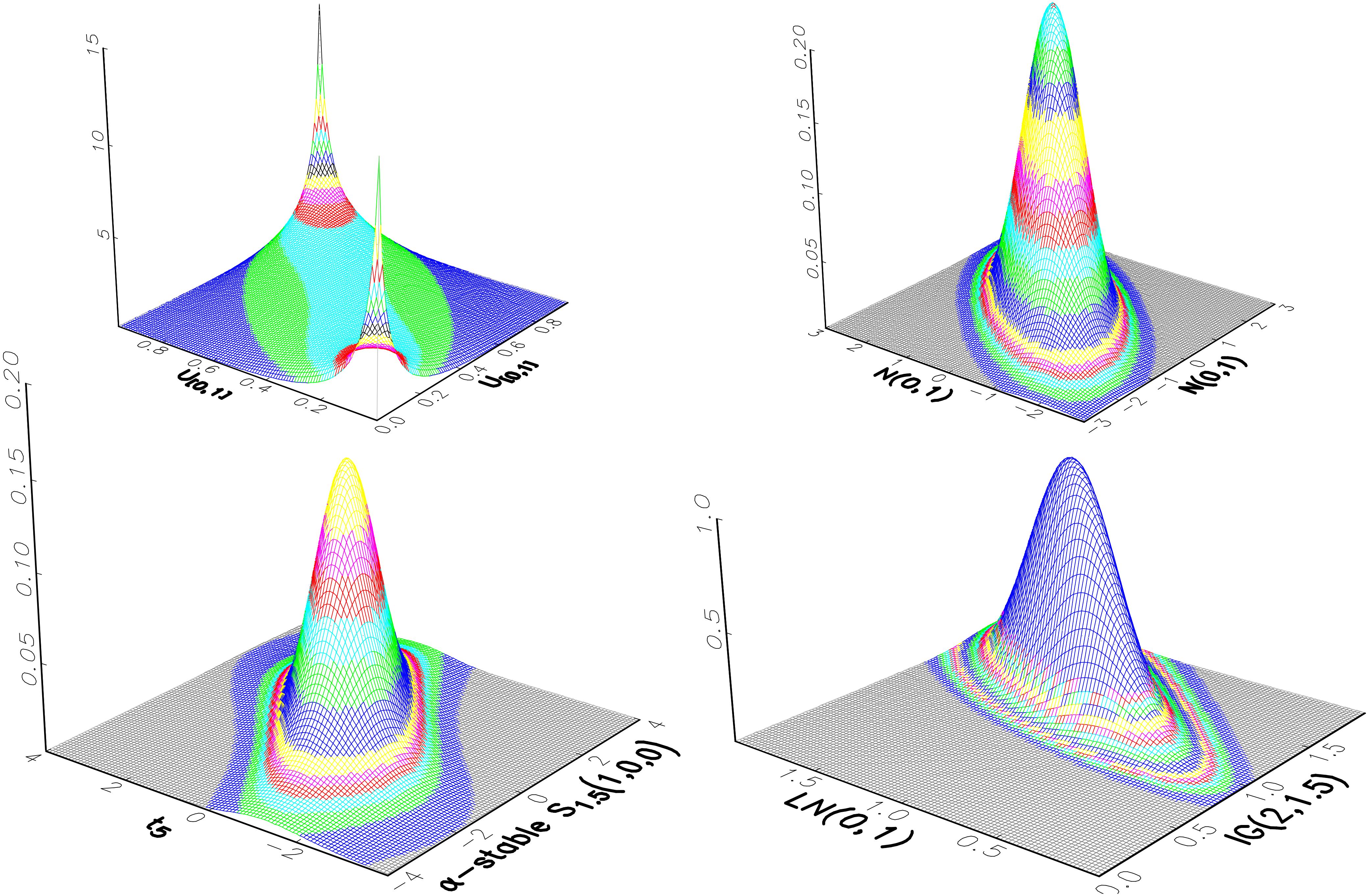
Because Normal copula is also the copula of log-normal random vector, it is the most used in finance.

The Normal copula satisfies

$$\mathbf{C}^- = \mathbf{C}_{-1} \prec \mathbf{C}_{\rho<0} \prec \mathbf{C}_0 = \mathbf{C}^\perp \prec \mathbf{C}_{\rho>0} \prec \mathbf{C}_1 = \mathbf{C}^+$$

It is a *comprehensive* copula ($\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$ are special cases of the Normal copula).

Is the set of Normal copulas $C\langle\mathcal{N}\rangle$ sufficient to characterize dependence in Finance ?

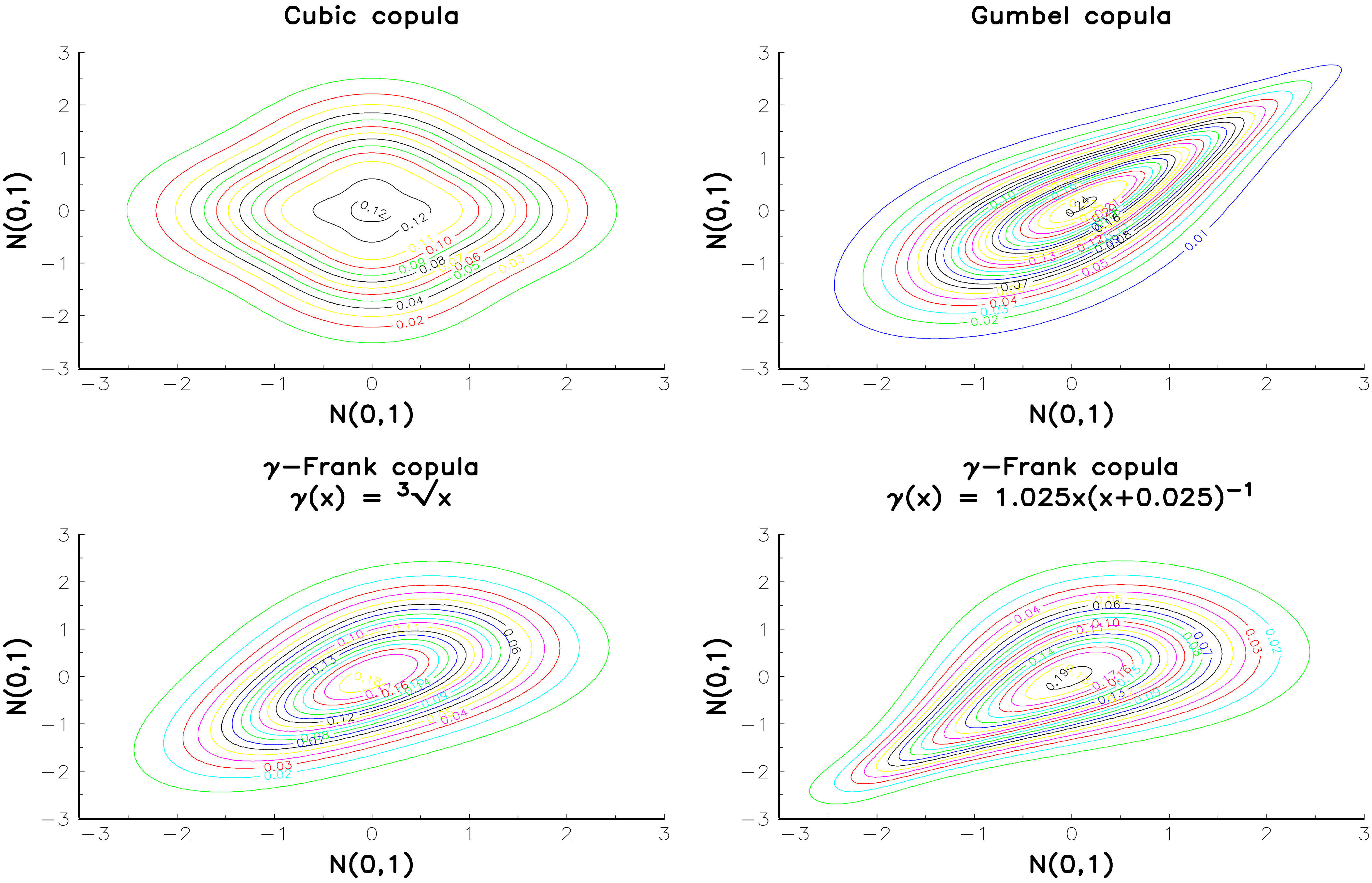


Normal Copula with τ = 0.5

4 Extreme and singular copulas

Some properties of the Normal copula:

- It is radially symmetric (bear market $\neq$ bull market);
- It is PQD or NQD;
- It is not an extreme value copula;
- It is not an LMP copula;
- It does not have a stochastic process representation mechanism.



4 Distributions with Gaussian Margins

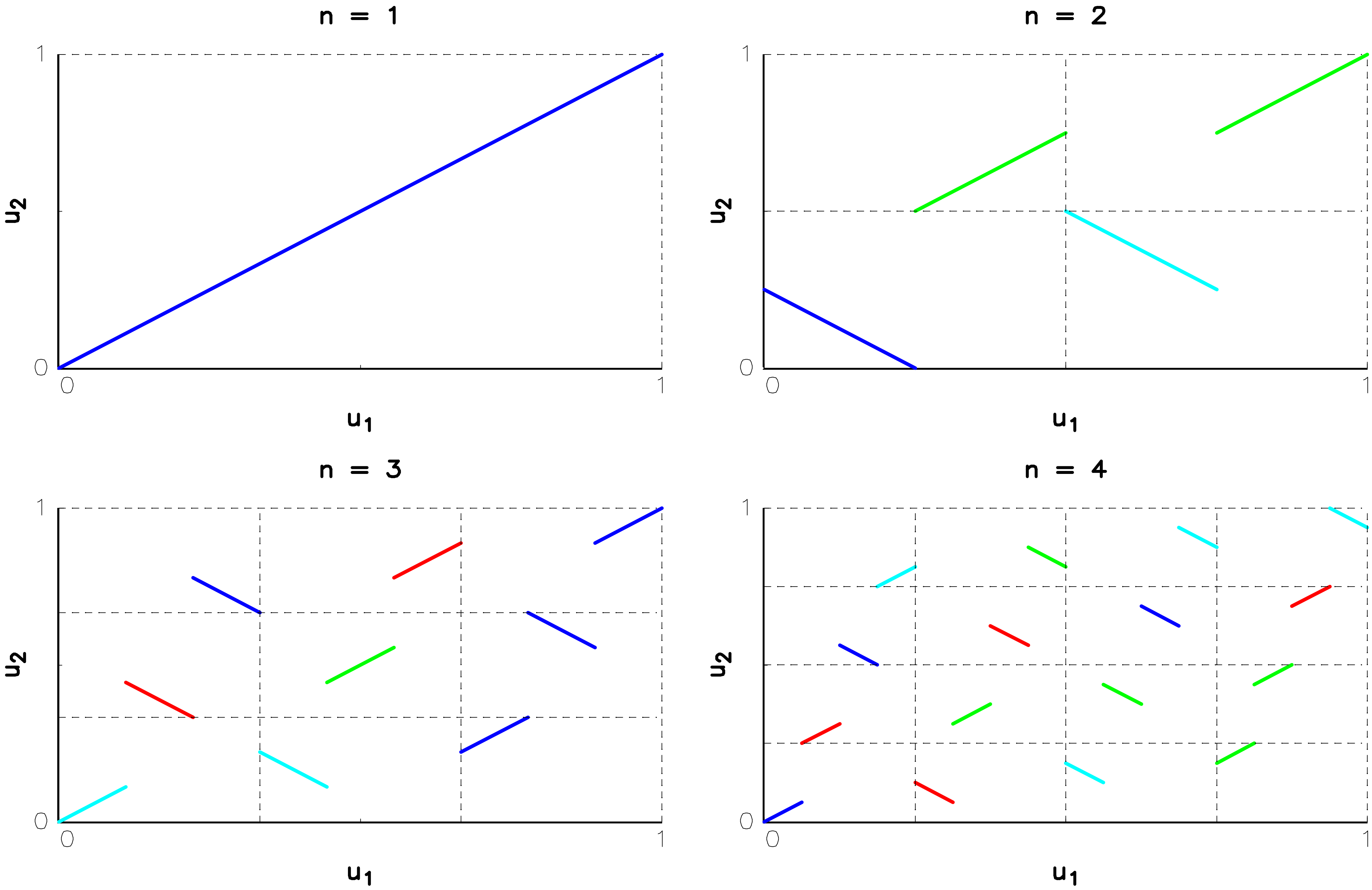
4.1 Singular copulas

Singular copulas do not have a density

$$\partial_{1,2}\mathbf{C}\left(u_1, u_2\right) = 0$$

- Singular copulas with prescribed support
- Ordinal sums of singular copulas
- Shuffles of Min

The mass distribution for a shuffle of Min can be obtained by (1) placing the mass for $\mathbf{C}^+$ on $[0,1]^2$, (2) cutting $[0,1]^2$ vertically into a finite number of strips, (3) shuffling the strips with perhaps some of them flipped around their vertical axes of symmetry, and then (4) reassembling them to form the square again. The resulting mass distribution will correspond to a copula called a shuffle of Min (Mikusiński, Sherwood and Taylor [1992]).



Shuffles of Min

4.2 Extreme copulas

The determination of the extreme points of $\mathcal{C}$ is an open problem (see the survey of Beneš and Štēpán [1991]).

The multiplication product of copulas has been defined by Darsow, Nguyen and Olsen [1992] in the following manner

$$\begin{array}{rcl} \mathbf{I}^2 & \longrightarrow & \mathbf{I} \\ (x,y) & \longmapsto & (\mathbf{C}_1 * \mathbf{C}_2)(x,y) = \int_0^1 \partial_2 \mathbf{C}_1(x,s)\, \partial_1 \mathbf{C}_2(s,y)\, \mathrm{d}s \end{array}$$

The transposition of copula corresponds to the mapping function $\mathbf{C}^\top(x,y) = \mathbf{C}(y,x)$.

Theorem 4 (Darsow, Nguyen and Olsen [1992]) *The set $\mathcal{C}$ is a symmetric Markov algebra under $*$ and $^\top$. The unit and null elements are $\mathbf{C}^\perp$ and $\mathbf{C}^+$.*

We say that $\mathbf{C}^{(-1]}$ (respectively $\mathbf{C}^{[-1)}$) is a *left* (*right*) inverse of $\mathbf{C}$ if $\mathbf{C}^{(-1]} * \mathbf{C} = \mathbf{C}^{+}$ ($\mathbf{C} * \mathbf{C}^{[-1)} = \mathbf{C}^{+}$).

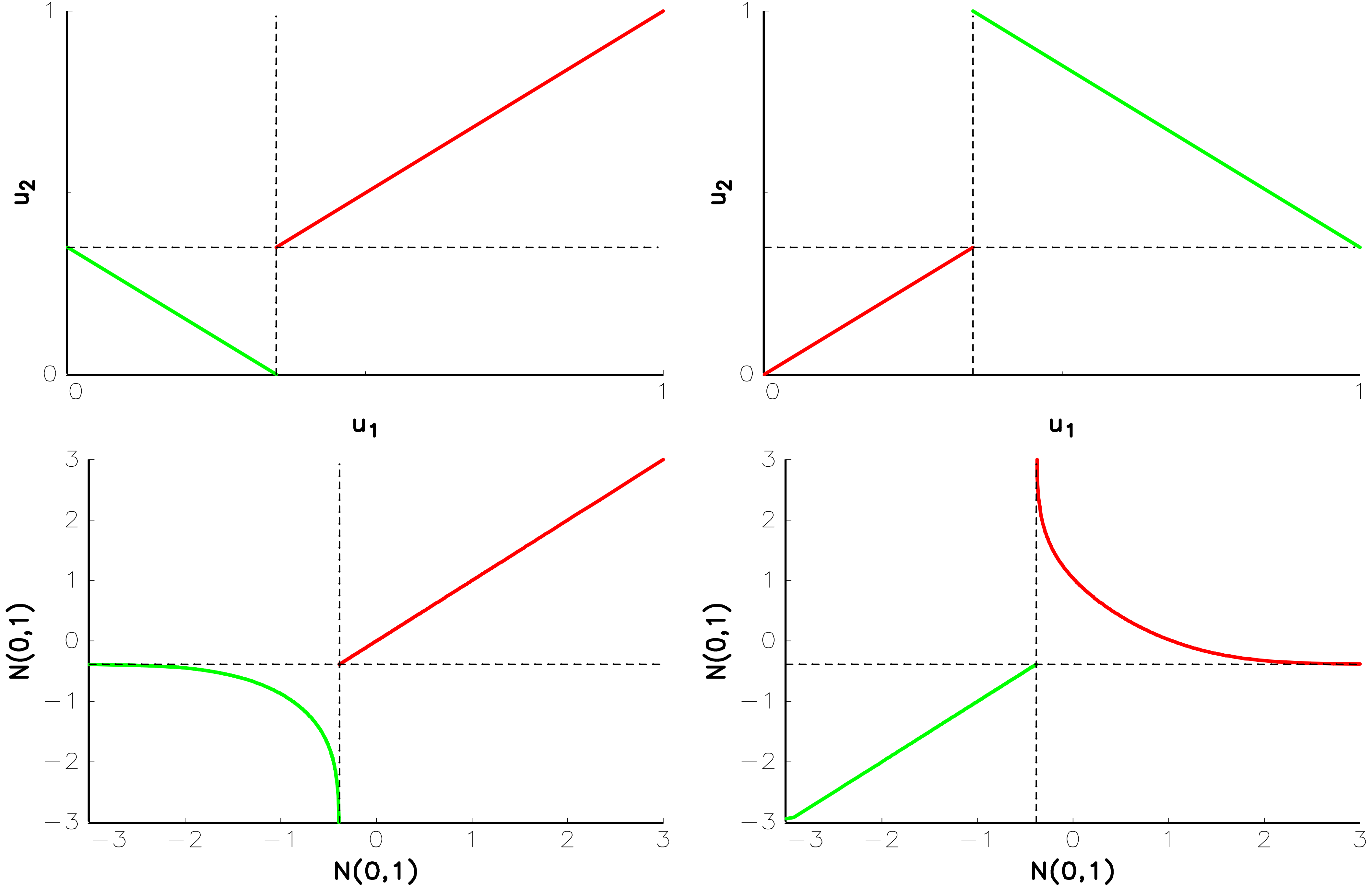
Theorem 5 (Darsow, Nguyen and Olsen [1992]) *Any element of $\mathcal{C}$ that possesses a left or right inverse is extreme.*

For example, $\mathbf{C}^{-} \in \mathcal{E}x\left(\mathcal{C}\right)$ because we have

$$\begin{aligned}
\left(\mathbf{C}^{-} * \mathbf{C}^{-}\right)(x,y) &= \int_0^1 \mathbf{1}_{[x+s-1\geq 0]} \mathbf{1}_{[s+y-1\geq 0]} \,\mathrm{d}s \\
&= \int_0^1 \mathbf{1}_{[s\geq 1-x]} \mathbf{1}_{[s\geq 1-y]} \,\mathrm{d}s \\
&= 1 - \min\left(1-x, 1-y\right) \\
&= \mathbf{C}^{+}(x,y)
\end{aligned}$$

It is easy to show that $\mathbf{C}^{+} \in \mathcal{E}x\left(\mathcal{C}\right)$ and $\mathbf{C}^{\perp} \notin \mathcal{E}x\left(\mathcal{C}\right)$.

$\Rightarrow$ Ordinal sums of $\mathbf{C}^{-}$ and $\mathbf{C}^{+}$ are extreme points.



Ordinal Sums of C^- and C^+

5 Quantile aggregation

Belief in finance:

$$\textbf{most important risks} \Leftrightarrow \rho = +1$$

Translation into the language of copulas:

$$\textbf{most risky dependence function} \Leftrightarrow \mathbf{C} = \mathbf{C}^{+}$$

This is true only for some financial problems (first-to-default, BestOf option, etc.).

$\Rightarrow$ **there are situations where most important risks do not correspond to the case $\rho = \rho^{+}$.**

5.1 Makarov inequalities

Let L denotes a two-place function (for example, the four arithmetic operators $+$, $-$, $\times$ and $\div$). The *supremal convolution* $\tau_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)$ is

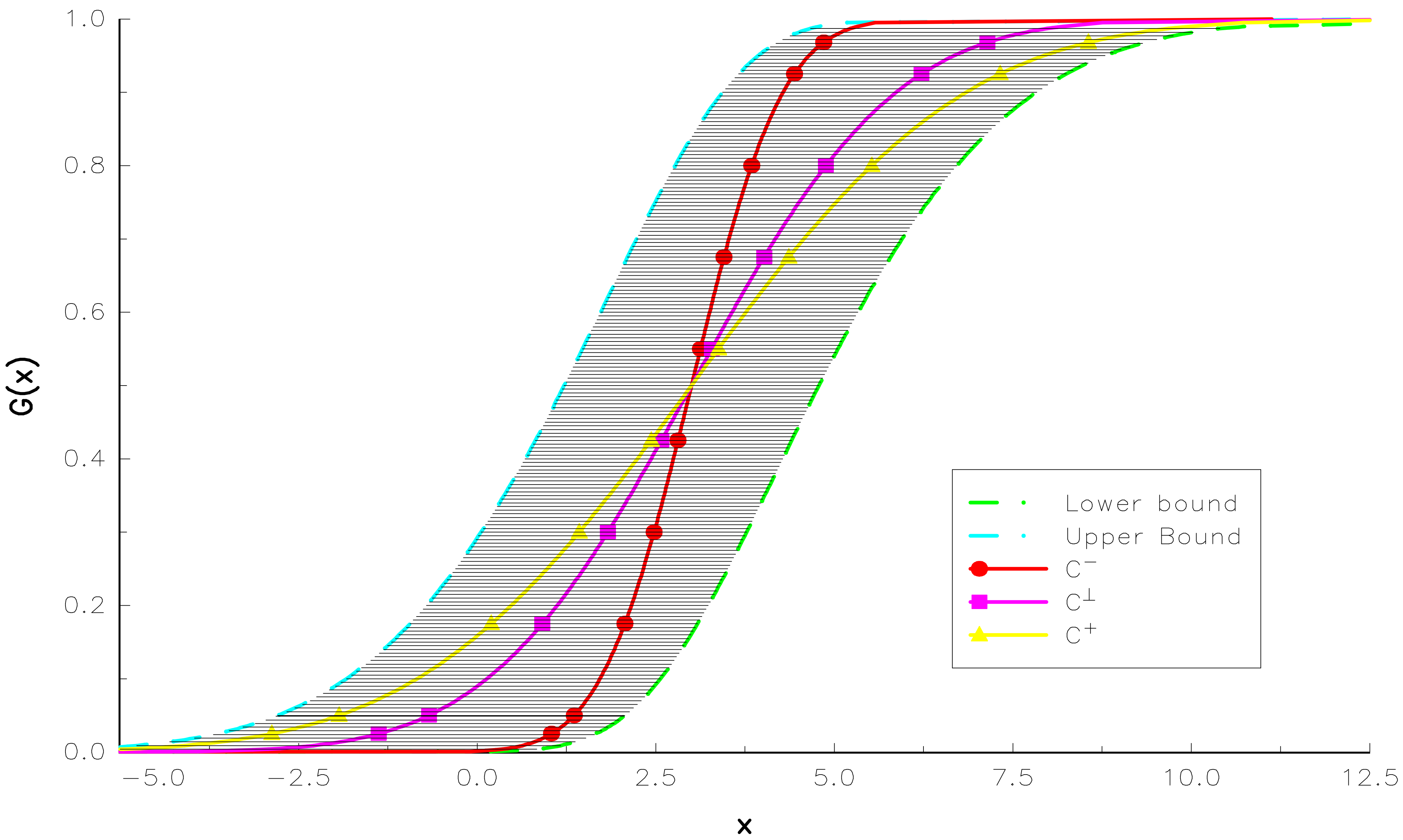
$$\tau_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)(x) = \sup_{L(x_1,x_2)=x} \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

whereas the *infimal convolution* $\rho_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)$ corresponds to

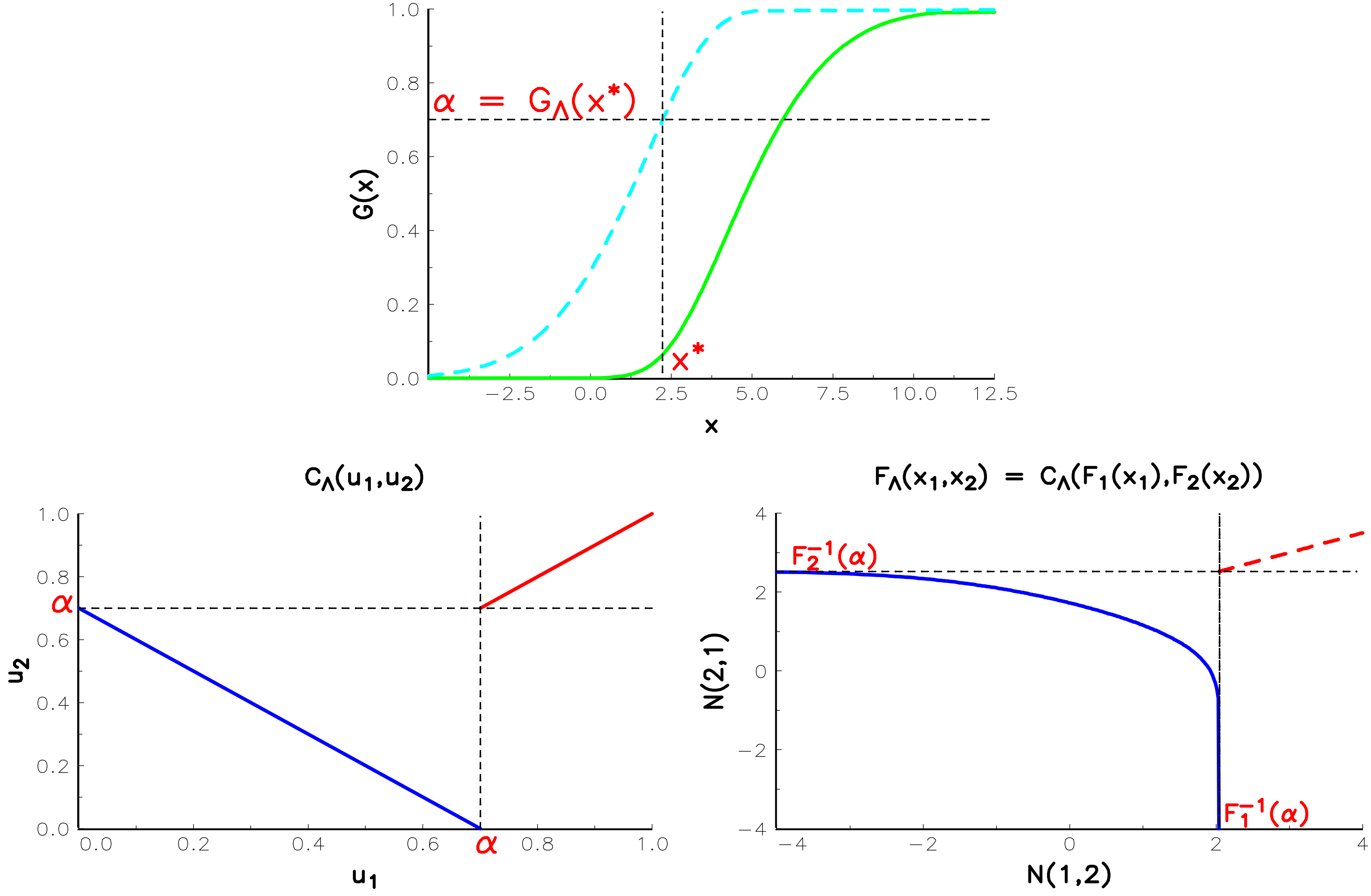
$$\rho_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)(x) = \inf_{L(x_1,x_2)=x} \tilde{\mathbf{C}}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

with $\tilde{\mathbf{C}}$ the dual of the copula $\mathbf{C}$ ($\tilde{\mathbf{C}}(u_1, u_2) = u_1 + u_2 - \mathbf{C}(u_1, u_2)$).

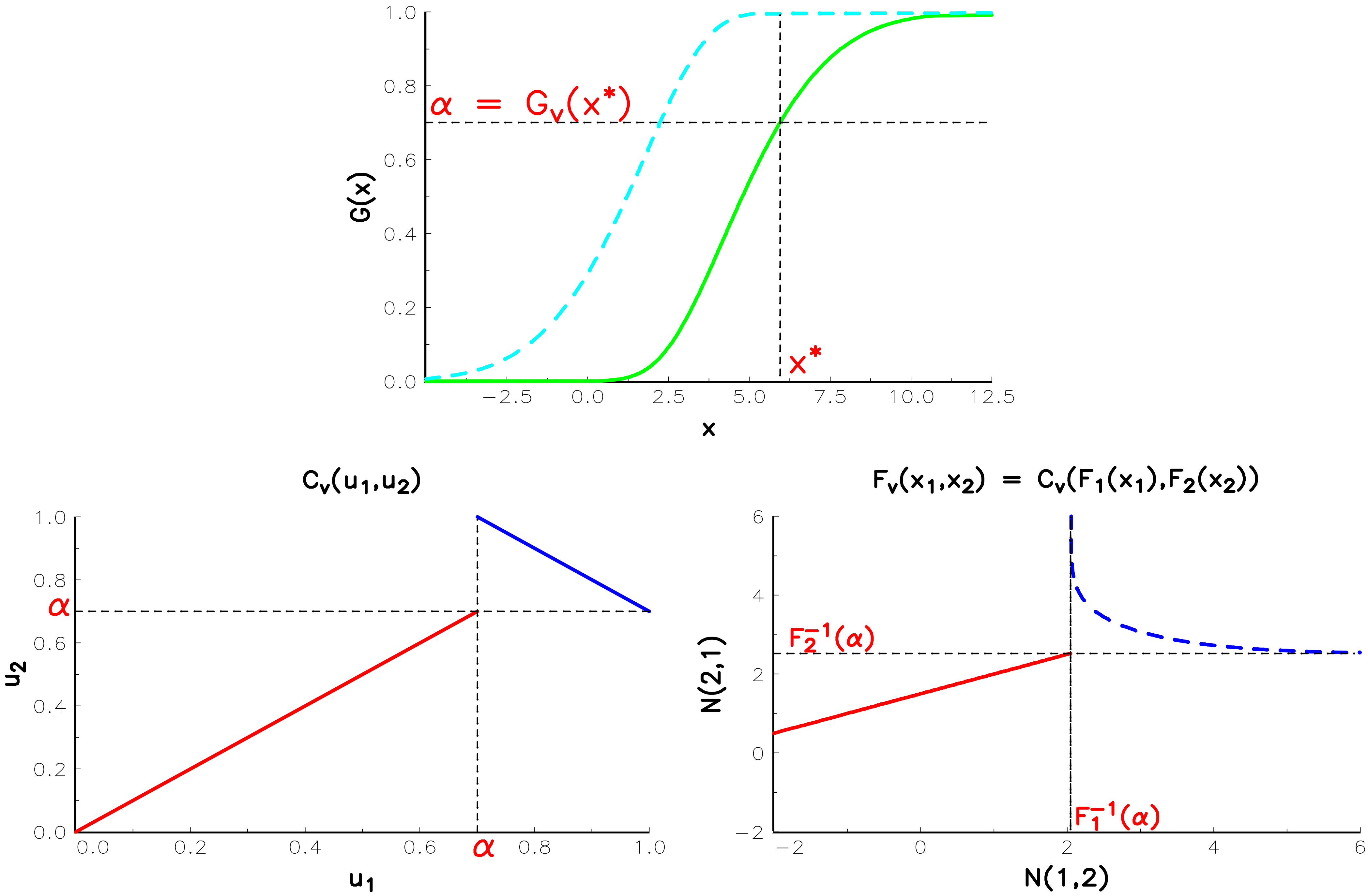
Frank, Nelsen and Schweizer [1987] and Williamson [1989] show that the distribution $\mathbf{G}$ of $X = L(X_1, X_2)$ is contained within the bounds $\mathbf{G}_\vee(x) \leq \mathbf{G}(x) \leq \mathbf{G}_\wedge(x)$ with $\mathbf{G}_\vee(x) = \tau_{\mathbf{C}_-,L}(\mathbf{F}_1, \mathbf{F}_2)(x)$ and $\mathbf{G}_\wedge(x) = \rho_{\mathbf{C}_-,L}(\mathbf{F}_1, \mathbf{F}_2)(x)$. **These bounds are the pointwise <u>best</u> possible.**



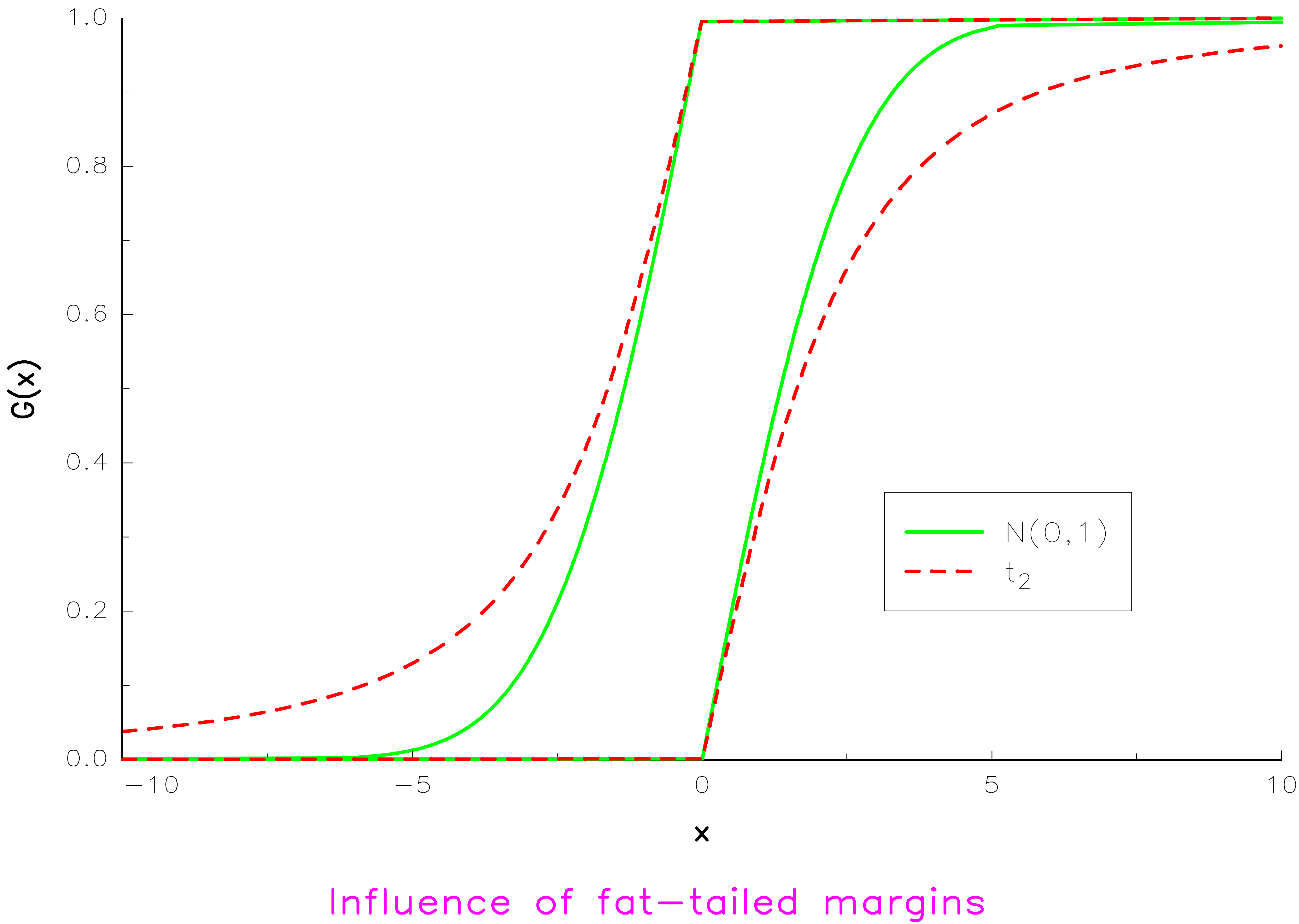
Dependency Bounds for the Convolution of random variables N(1,2) and N(2,1)



Upper Dependency Bound



Lower Dependency Bound



1.0
0.8
0.6
0.4
0.2
0.0
G(x)
−10
−5
0
5
10
x
N(0,1)
t2
Influence of fat−tailed margins

5.2 Dependency bounds of the VaR

Using the **duality** theorem of Frank and Schweizer [1979], it comes that if $\mathbf{C}_{-} = \mathbf{C}^{-}$ and L is the operation $+$, we have

$$\mathbf{G}_{\vee}^{(-1)}(u) = \inf_{\max(u_1+u_2-1,0)=u} \mathbf{F}_1^{(-1)}(u_1) + \mathbf{F}_2^{(-1)}(u_2)$$

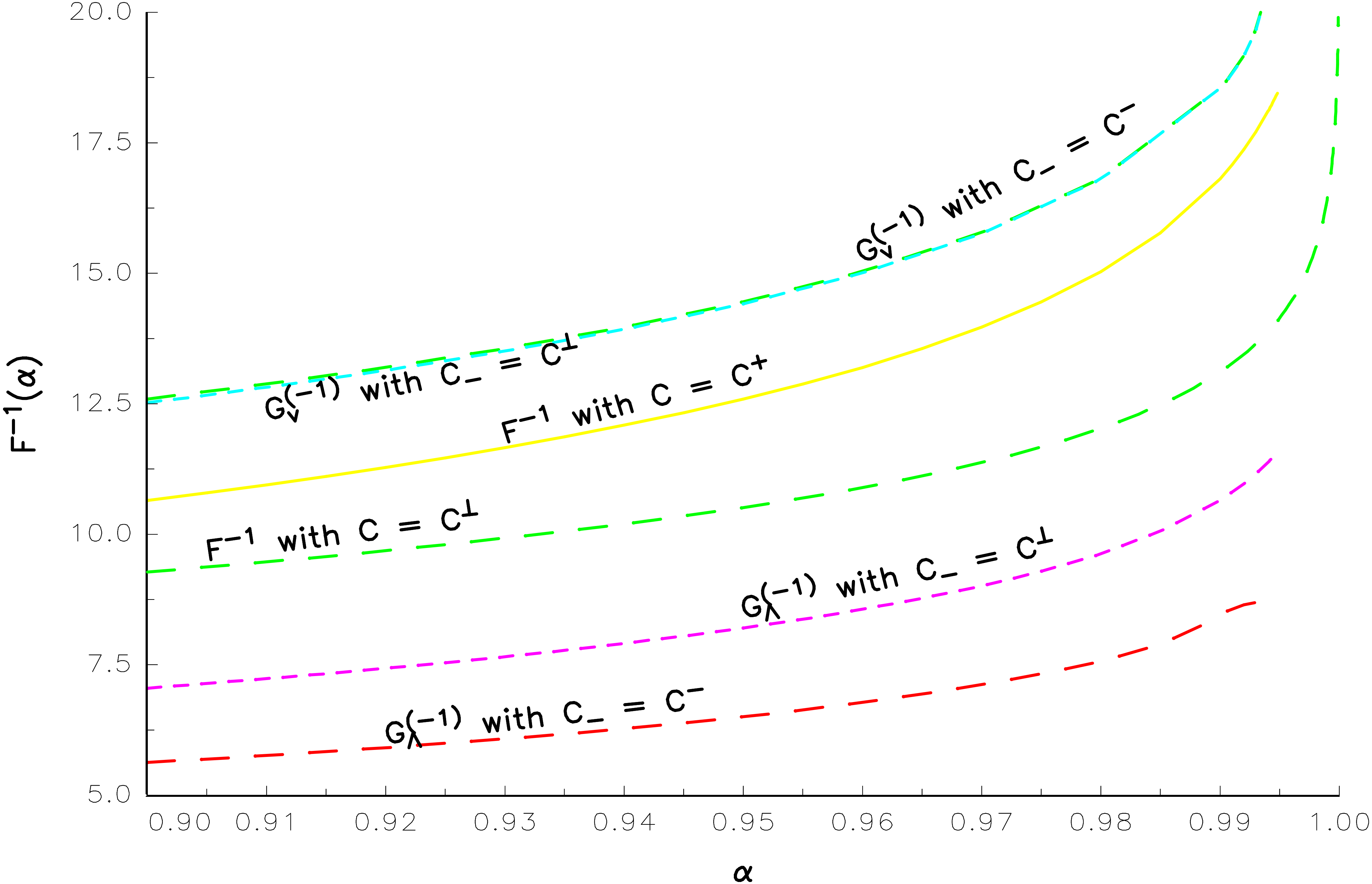
and

$$\mathbf{G}_{\wedge}^{(-1)}(u) = \sup_{\min(u_1+u_2,1)=u} \mathbf{F}_1^{(-1)}(u_1) + \mathbf{F}_2^{(-1)}(u_2)$$

We recall that $\mathrm{VaR}_{\alpha}(X) = \mathbf{F}^{-1}(\alpha)$. The corresponding dependency bounds are then

$$\mathbf{G}_{\wedge}^{(-1)}(\alpha) \leq \mathrm{VaR}_{\alpha}(X_1 + X_2) \leq \mathbf{G}_{\vee}^{(-1)}(\alpha)$$

Numerical algorithms to compute the dependency bounds exist (for example the *uniform quantisation* method of Williamson [1989]).



Dependency bounds for VaR with Gamma margins

5.3 The diversification effect

If we define the diversification effect as follows

$$\mathsf{D} = \frac{\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) - \mathrm{VaR}_\alpha(X_1 + X_2)}{\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)}$$

there are situations where $\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$. A more appropriate definition is then

$$\bar{\mathsf{D}} = \frac{\mathbf{G}_{\vee}^{(-1)}(\alpha) - \mathrm{VaR}_\alpha(X_1 + X_2)}{\mathbf{G}_{\vee}^{(-1)}(\alpha)}$$

Embrechts, McNeil and Straumann [1999] interpret $\chi\left(\mathbf{C}_{\vee}^{(\alpha)}, \mathbf{C}^{+}; \alpha\right)$

$$\chi\left(\mathbf{C}_{\vee}^{(\alpha)}, \mathbf{C}^{+}; \alpha\right) = \frac{\mathbf{G}_{\vee}^{(-1)}(\alpha) - \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)}{\mathbf{G}_{\vee}^{(-1)}(\alpha)}$$

as "*the amount by which VaR fails to be subadditive*".

5.4 VaR aggregation in practice

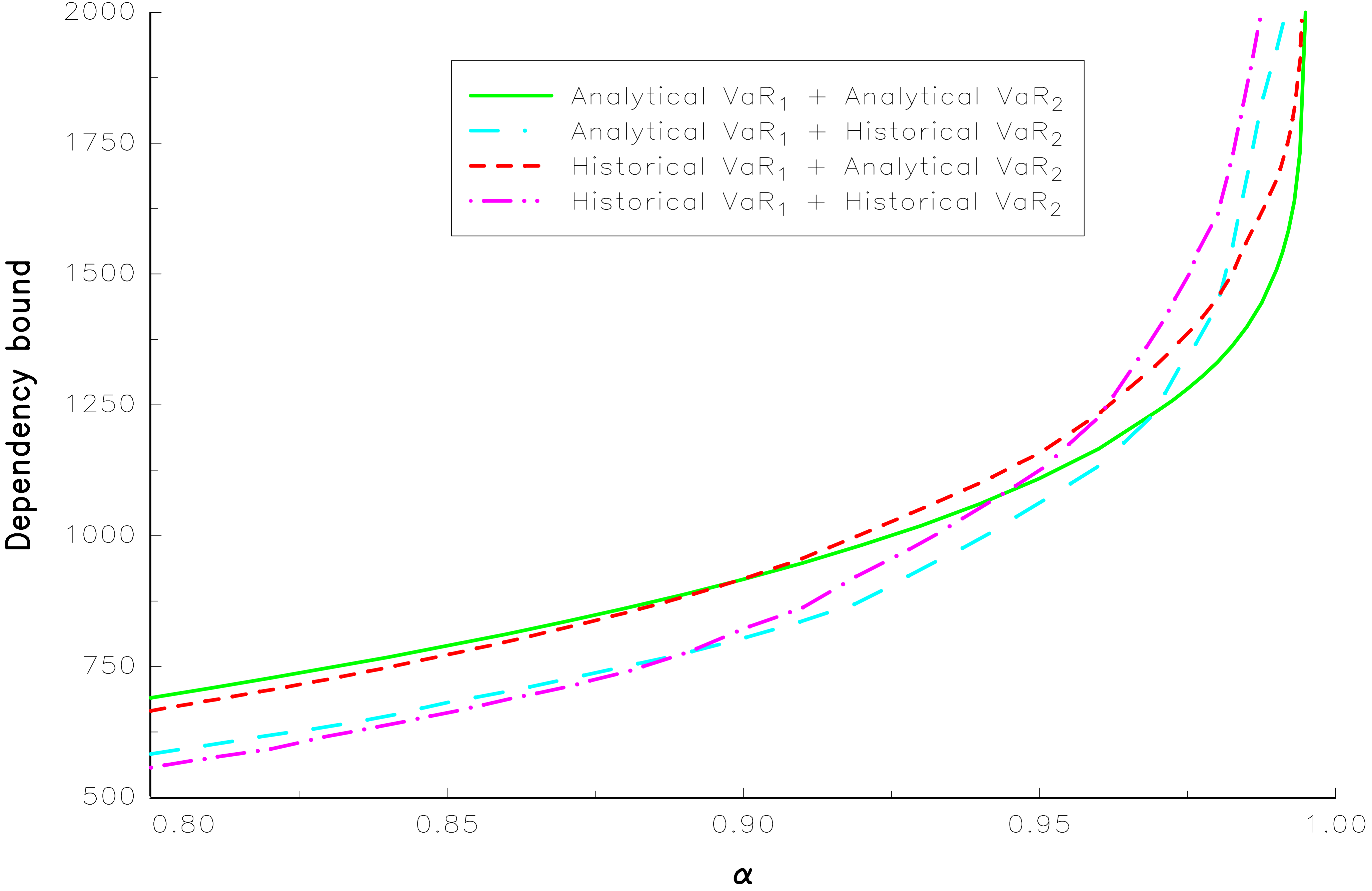
LME example of Durrleman, Nickeghbali and Roncalli [2000]:

	AL	AL-15	CU	NI	PB
P_1	5	3			
P_2			5	2	−3

	Analytical *VaR*	Historical *VaR*
P_1	363.05	445.74
P_2	1026.03	1274.64

Here are the values of $\mathbf{G}_{\vee}^{(-1)}(\alpha)$ for α equal to 99%:

	P_1 Analytical *VaR*	P_1 Historical *VaR*
P_2 Analytical *VaR*	1507.85	1680.77
P_2 Historical *VaR*	1930.70	2103.67



Dependency bounds for VaR with LME example

6 Application to two assets option pricing

What is a conservative correlation ?

$\Downarrow$

What is a conservative dependence function ?

6.1 Multivariate RNDs and copulas

Let $\mathbb{Q}_n$ and $\mathbb{Q}$ be the risk-neutral probability distributions of $S_n(T)$ and $\mathbf{S}(T) = \begin{pmatrix} S_1(T) & \cdots & S_N(T) \end{pmatrix}^\top$. With arbitrage theory, we can show that

$$\mathbb{Q}(+\infty, \ldots, +\infty, S_n(T), +\infty, \ldots, +\infty) = \mathbb{Q}_n(S_n(T))$$

$\Rightarrow$ The margins of $\mathbb{Q}$ are the RNDs $\mathbb{Q}_n$ of Vanilla options.

Breeden et Litzenberger [1978] remark that European option prices permit to caracterize the probability distribution of $S_n(T)$

$$\begin{aligned} \phi(T, K) &:= 1 + e^{r(T-t_0)} \frac{\partial C(T, K)}{\partial K} \\ &= \Pr\{S_n(T) \leq K\} \\ &= \mathbb{Q}_n(K) \end{aligned}$$

Durrleman [2001] extends this result in the bivariate case:

1. for a call max option, $\phi(T,K)$ is the diagonal section of the copula

$$\phi(T,K) = \mathbf{C}(\mathbb{Q}_1(K), \mathbb{Q}_2(K))$$

2. for a spread option, we have

$$\phi(T,K) = \int_0^{+\infty} \partial_1 \mathbf{C}(\mathbb{Q}_1(x), \mathbb{Q}_2(x+K))\, \mathrm{d}\mathbb{Q}_1(x)$$

$\Rightarrow$ Other results are derived in Durrleman [2001] (bounds, general pricing kernel, etc.) — see Coutant, Durrleman, Rapuch and Roncalli [2001].

6.2 Computation of the implied parameter $\widehat{\rho}$

- BS model: LN distribution calibrated with ATM options; Pricing kernel = LN distributions + Normal copula

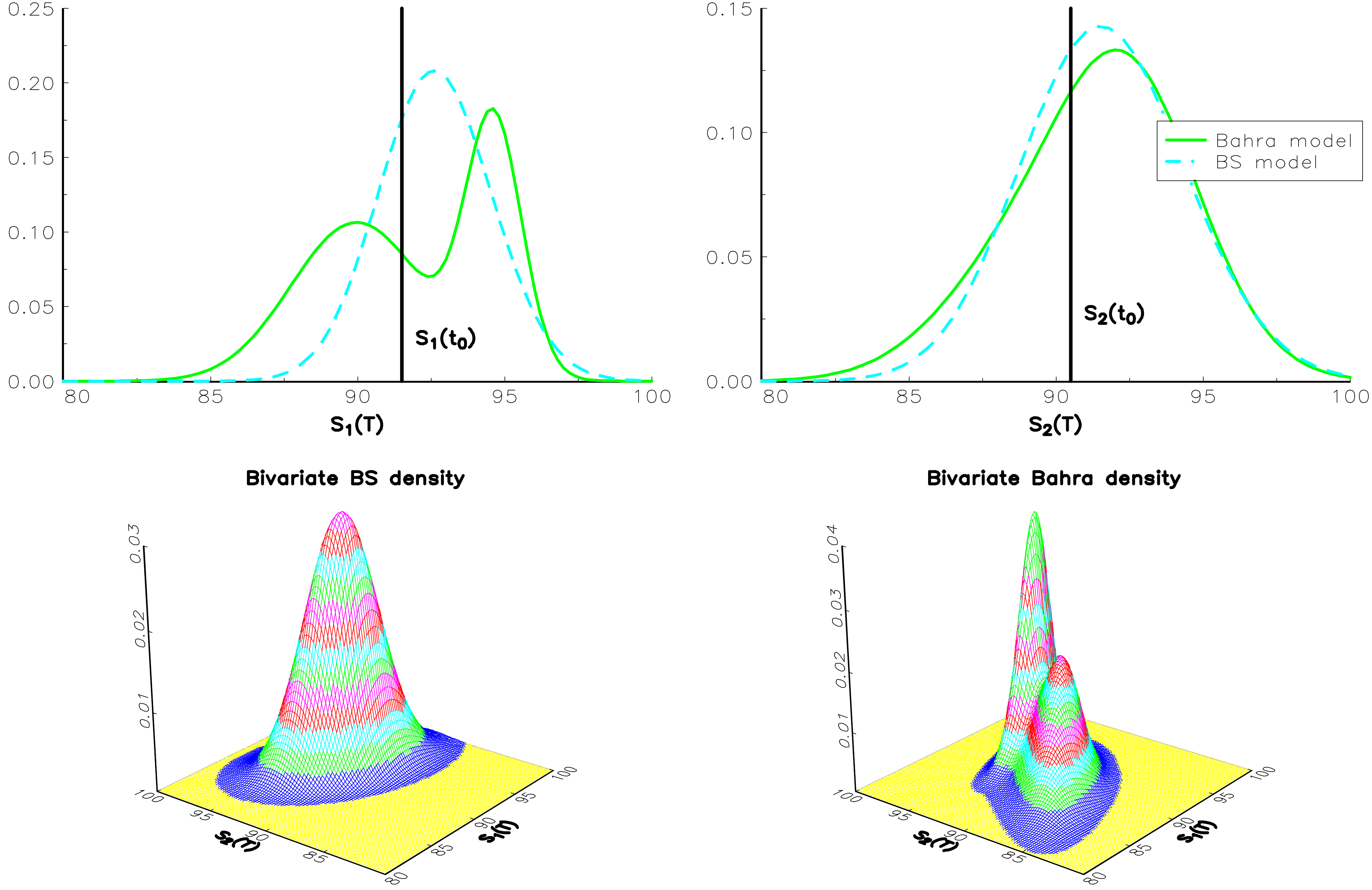
$$\widehat{\rho}_1 = -0.341$$

- Bahra model: mixture of LN distributions calibrated with eight European prices; Pricing kernel = MLN distributions + Normal copula

$$\widehat{\rho}_2 = 0.767$$

Remark 1 *$\widehat{\rho}_1$ and $\widehat{\rho}_2$ are parameters of the Normal Copula. $\widehat{\rho}_1$ is a Pearson correlation, not $\widehat{\rho}_2$.*

$\Rightarrow$ BS model: negative dependence / Bahra model: positive dependence.



A spread option example

6.3 Bounds of a spread option

For some two-assets options, bounds are related to Fréchet copulas (see Cherubini and Luciano [2000] for binary options and Coutant, Durrleman, Rapuch and Roncalli [2001] for BestOf/WorstOf options).

For spread options, bounds are more complicated, but can be related to Vanilla prices. For example, we obtain when $K > 0$

$$\int_0^K \sup_{u \geq x} \left(\partial_K C_1(T, u - x) - \partial_K C_2(T, u)\right)^+ \, \mathrm{d}x \leq K e^{-rT} - \mathrm{CS}\,(T, 0) + \mathrm{CS}\,(T, K)$$

$$K e^{-rT} - \mathrm{CS}\,(T, 0) + \mathrm{CS}\,(T, K) \leq K e^{rT} - \int_0^K \sup_{u \geq x} \left(\partial_K C_1(T, u - x) - \partial_K C_2(T, u)\right)$$

$\Rightarrow$ **What is a conservative dependence function ?**

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