Option hedging with stochastic volatility

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Abstract

The purpose of this paper is to analyse different implications of the stochastic behavior of asset prices volatilities for option hedging purposes. We present a simple stochastic volatility model for option pricing and illustrate its consistency with financial stylized facts. Then, assuming a stochastic volatility environment, we study the accuracy of Black and Scholes implied volatility-based hedging. More precisely, we analyse the hedging ratios biases and investigate different hedging schemes in a dynamic setting.

1 Introduction

Assumptions concerning the underlying asset price dynamics are the fundamental characteristic of any option-pricing model. The classical BLACK and SCHOLES [1973] model assumes that the asset price is generated by a geometric Brownian motion. However, many empirical studies document the excess kurtosis of financial asset returns' distributions and their conditional heterockedasticity.

Models that allow the volatility of asset prices to change randomly are consistent with these observations. Moreover, a stochastic volatility environment justifies the existence of the smile effect for the Black and Scholes implied volatilities and permits us to explain its features. Traditionally, the stochastic behavior of volatility was explained in terms of information arrivals or related to changes in the level of the stock price (see CHRISTIE [1982]).

Despite its strong empirical rejection, the Black and Scholes model is commonly used by practitioners, often in an internally inconsistent manner. For example, the hedging properties of the Black and Scholes model seem better when one uses the series of implied volatilities rather than the close-to-close historical volatility data. In this way, one admits that asset returns' variability changes over time and one uses a model that assumes a constant volatility diffusion process for the asset prices.

The reason of popularity of the Black and Scholes model is its simplicity. It gives a simple formula for the option price and permits us to explicitly calculate option hedging ratios. On the other hand, a stochastic volatility option pricing model requires the use of numerical techniques for option price and greeks computing. Moreover, its practical implementation requires a preliminary estimation of the parameters of the unobservable latent volatility process (see GHYSELS and al. [1995] for a survey on this topic).

HESTON [1993] derived a closed-form solution for European options in a special stochastic volatility environment. Generally, researchers have used Monte Carlo or finite difference methods to solve stochastic volatility option pricing problems. KURPIEL and RONCALLI [1998] show how to apply Hopscotch methods, a class of finite difference algorithms introduced initially by GOURLAY [1970], to two-state financial models. Unlike Monte Carlo, Hopscotch methods are very useful for American option pricing and easy greeks computing in a stochastic volatility framework.

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The object of this paper is to investigate different implications of the stochastic behavior of volatilities for option hedging purposes. We analyse different hedging strategies and study the accuracy of Black and Scholes methods when volatilities of asset prices are random. Previous work concerned with the performance of hedging schemes for options on stocks has been carried out by BOYLE and EMANUEL [1980] and GALAI [1983]. Boyle and Emanuel look into the distribution of discrete rebalanced delta hedge cost in the Black and Scholes world. Galai studies the sources of the cost arising from the discrete delta hedging. HULL and WHITE [1987 b] analyse the performance of different hedging schemes for currency options in a simple stochastic volatility environment. However, they use the Black and Scholes formula to approximate option prices and hedging ratios. Moreover, they concentrate only on the case where the asset price returns are not correlated with their volatilities.

The paper is organized as follows. In section 2, we briefly present a stochastic volatility model for option pricing. Then, in section 3, we confront it with some financial stylized facts. Finally, in section 4, we discuss option hedging problems in stochastic volatility environment.

2 Stochastic volatility model for option pricing

A stochastic volatility option pricing model is a special case of the two-state financial model, with two sources of risk. The two-dimensional state vector $X(t) = \begin{bmatrix} S(t) & \sigma(t) \end{bmatrix}^\top$ is generated by a diffusion defined from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is the fundamental space of the underlying asset price process $S(t)$

$$\begin{bmatrix} dS(t) \\ d\sigma(t) \end{bmatrix} = \begin{bmatrix} \mu S(t) \\ \mu_2(t, S(t), \sigma(t)) \end{bmatrix} dt + \begin{bmatrix} \sigma(t) S(t) & \sigma_{1,2}(t, S(t), \sigma(t)) \\ \sigma_{2,1}(t, S(t), \sigma(t)) & \sigma_{2,2}(t, S(t), \sigma(t)) \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \quad (1)$$

with $E[W_1(t) W_2(t)] = \rho t$. The risk-free interest rate $r(t)$ is assumed constant or deterministic. The market permits continuous and frictionless trading and no arbitrage opportunities exists. However, because there is no asset that is clearly instantaneously perfectly correlated with the state variable $\sigma(t)$, the market is not complete.

In this case, assuming that $\sigma_{1,2}(t, S(t), \sigma(t)) = \sigma_{2,1}(t, S(t), \sigma(t)) = 0$, the valuation partial differential equation for a contingent claim $P(t)$ on an asset paying a continuous dividend $d = d(t, S(t), \sigma(t))$ reduces with simplified notation to

$$\begin{cases} \frac{1}{2}\sigma^2(t) S^2(t) P_{SS} + \frac{1}{2}\sigma_{2,2}^2 P_{\sigma\sigma} + \rho\sigma(t) S(t) \sigma_{2,2} P_{S\sigma} \\ + [rS(t) - d] P_S + [\mu_2 - \lambda_2(t) \sigma_{2,2}] P_\sigma + P_t - rP = 0 \\ P(T) = f(S(T), \sigma(T)) \end{cases} \quad (2)$$

where $\lambda_2(t)$ is called the volatility risk premium process. For any choice of $\lambda_2(t)$, the solution of the system (2) is an admissible price process for the contingent claim $P(t)$.

HESTON [1993] suggests an Ornstein-Uhlenbeck process for the volatility which by application of Ito's lemma leads to a square-root process for the instantaneous variance $v(t) = \sigma^2(t)$. The state diffusion process becomes

$$\begin{bmatrix} dS(t) \\ dv(t) \end{bmatrix} = \begin{bmatrix} \mu S(t) \\ \kappa[\theta - v(t)] \end{bmatrix} dt + \begin{bmatrix} \sqrt{v(t)} S(t) & 0 \\ 0 & \sigma_v\sqrt{v(t)} \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \quad (3)$$

In BREEDEN's [1979] intertemporal asset pricing model, the assets risk premia are proportional to their instantaneous covariance with respect to aggregate consumption growth. In this case

$$\lambda_2(t) = \gamma \text{Cov}\left[dv(t), \frac{dC(t)}{C(t)} \mid \mathcal{F}_t\right] \quad (4)$$

where $\gamma = -C\frac{U_{CC}}{U_C}$ is the constant relative risk aversion coefficient. Considering the consumption process that emerges in the general equilibrium model of COX, INGERSOLL, and ROSS [1985]

$$dC(t) = \mu_c v(t) C(t) dt + \sigma_c \sqrt{v(t)} C(t) dW_3(t) \quad (5)$$
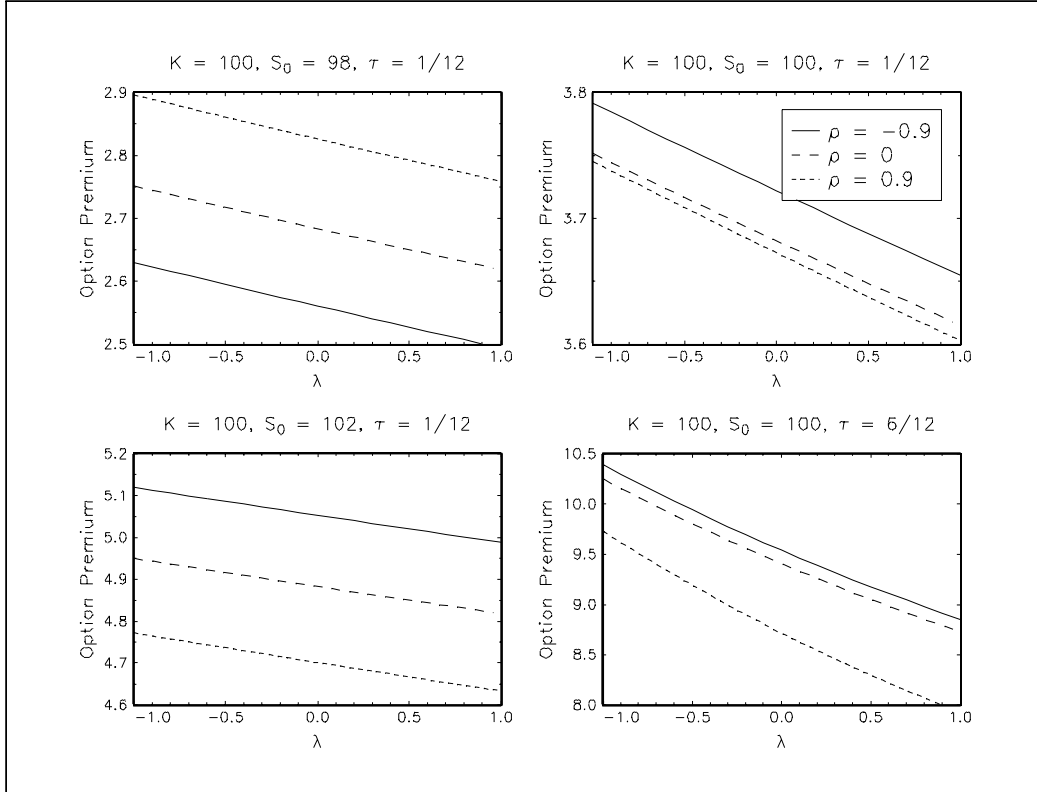
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Figure 1: Influence of the volatility risk premium on the European call option prices

and assuming that $E[W_2(t)W_3(t)] = \rho^\star t$, Heston generates the volatility risk premium that is proportional to the current value of the instantaneous variance process

$$\lambda_2(t) = \lambda v(t) \tag{6}$$

with $\lambda = \gamma\sigma_c\sigma_v\rho^\star$.[1]

In such a framework, the contingent claim price $P(t)$ becomes a linear function of λ. Consequently, the choice of λ is irrelevant for option hedging purposes and in the remainder of this paper we set $\lambda = 0$ for simplicity. This assumption states that the volatility risk premium is noncompensated.

Setting $\lambda = 0$ is consistent with the log utility function of the representative agent ($\gamma = 0$) or with a zero correlation between the volatility and the aggregate consumption growth. In the latter case, the price fluctuations due to the random term in the variance are completely diversifiable : the volatility has zero systematic risk. The assumption of the zero volatility risk premium was interpreted by FOLLMER and SCHWEIZER [1992] as the choice of an equivalent martingale measure which is closest to $\mathbb{P}$ in terms of relative entropy.

3   Stochastic volatility and some financial stylized facts

The ability to reproduce empirical stylized facts is an important model specification and selection criterion. In this section, we consider the stochastic volatility data generating process as defined by the diffusion (3).

We choose the following parameter values : $\mu = 0$, the long-run instantaneous variance mean $\theta = 0.3^2$, the mean-reversion speed $\kappa = 0.5$, the volatility of the variance process $\sigma_v = 0.9$, the constant risk-free

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[1] This is a little stronger restriction that the Assumption 2.2 in RENAULT and TOUZI [1996].
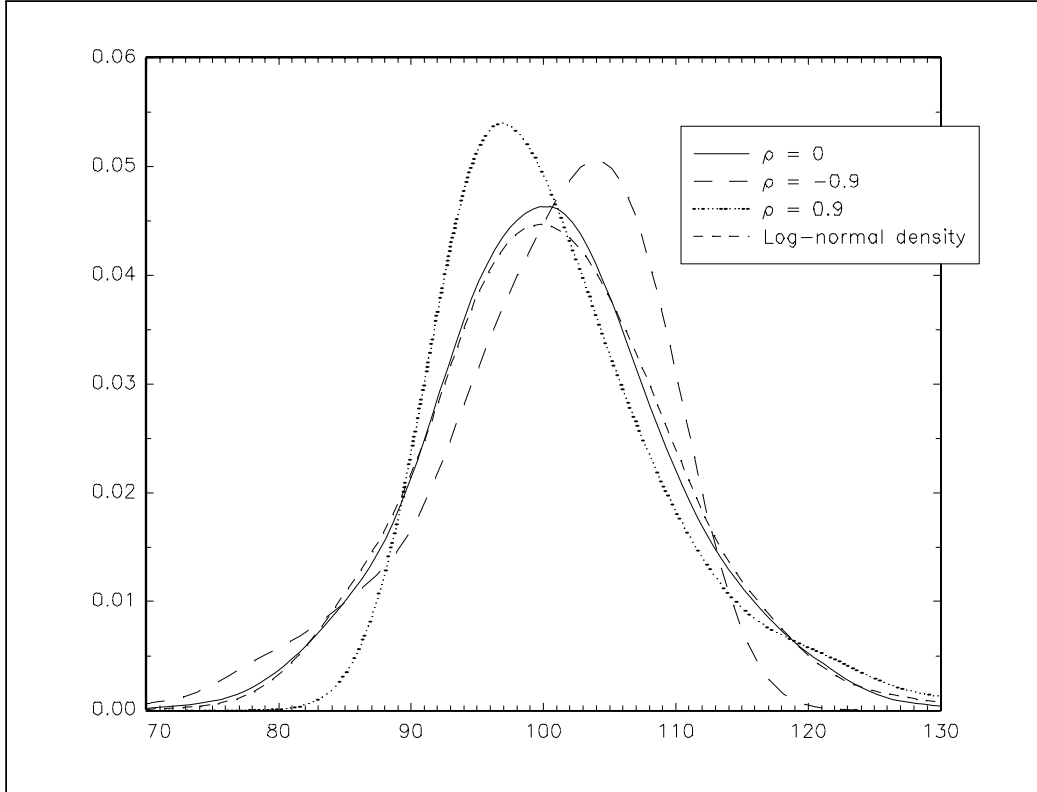
Figure 2: Terminal stock price distributions

instantaneous interest rate $r = 0.08$. We assume that the current variance $v(t_0)$ is equal to its long-run mean θ.

Fixing the time horizon to one month $\tau = \frac{1}{12}$, we simulated our stochastic volatility model 100 times with 2000 discretisation points. We find the following average skewness and kurtosis statistics for the asset price returns :

	Sk	t(H0: Sk = 0)	Ku	t(H0: Ku = 3)
$\rho = 0$	0.000	0.014	3.454*	4.147
$\rho = -0.9$	0.034	0.624	3.377*	3.444
$\rho = 0.9$	-0.031	0.566	3.393*	3.588

The parameter ρ corresponds to the instantaneous correlation of random shocks that affect the asset price and its volatility process.

BLACK [1976] suggests that stock price movements are negatively correlated with volatility. Because falling stock prices imply an increased leverage of firms, this entails more uncertainty and hence the stock price volatility tends to rise. In a simple Modigliani/Miller world with no dividends and a constant interest rate, the assumption of $\rho = 0$ is equivalent to a constant value of the firm or a zero leverage. JOHNSON and SHANNO [1987] suggest that if the firm's profitable investments tend to have high variance, then the continuous stream of news about these investments will produce increments in stock price and its volatility, which are positively correlated. Similarly, if the new investments have low variance, then ρ should be negative. However, SCOTT [1991] by studying transactions prices from CBOE was unable to compute estimates of ρ significantly different from zero. He worked in a stochastic volatility framework where the log-volatility follows an Orstein-Uhlenbeck process.

The table above shows that stochastic volatility implies heavy tails in the distribution of stock price returns. ARCH models provide a formal link between dynamic conditional volatility behavior and un-
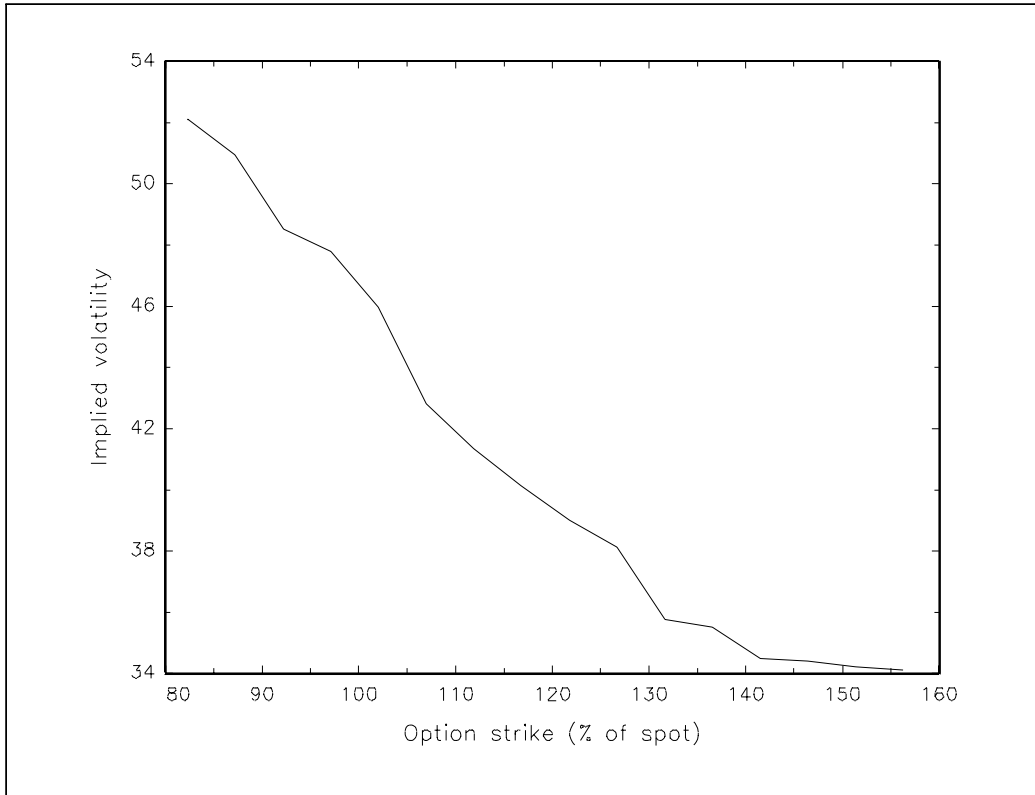
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Figure 3: Smile curve for European call options on the CAC 40 on October 2, 1998

conditional leptokurtosis. The skewness effects presented in the table are consistent with the sign of ρ but statistically insignificant. In fact, empirical evidence reported by BLACK [1976], CHRISTIE [1982] and SCHWERT [1989] suggests that leverage alone is too small to explain the empirical asymmetries one observes in stock prices. The parameter ρ corresponds to the current correlation between stock return's and its variance's shocks, when skewness of asset price returns is essentially characterized by the negative correlation between the current stock price and its *future* volatility.

On the other hand, the impact of ρ on the terminal stock price $S\left(T\right)$ distribution is clear. The shape of the terminal stock price distribution explains Black and Scholes pricing biases caused by stochastic volatility. For example, when $\rho < 0$, Black and Scholes model tends to overestimate the price of out-of-the-money (underestimate the price of in-the-money) call options and underestimate the price of out-of-the-money (overestimate the price of in-the-money) put options. This is because when the stock price increases, volatility tends to decrease, making it less likely that really high stock prices will be achieved. When the stock price decreases, volatility tends to increase, making it more likely that really low stock prices will be achieved. The same pricing biases are well reflected in the smile effect implied by a stochastic character of the volatility.

The Black and Scholes implied (or implicit) volatility is defined as the value of σ which equates the option price given by the Black and Scholes formula to the observed market price of the option. If option prices in the market were conformable with the Black and Scholes model, all the Black and Scholes implied volatilities corresponding to various options written on the same asset would coincide with the volatility parameter σ of the underlying asset. In reality this is not the case, and the Black and Scholes implied volatility heavily
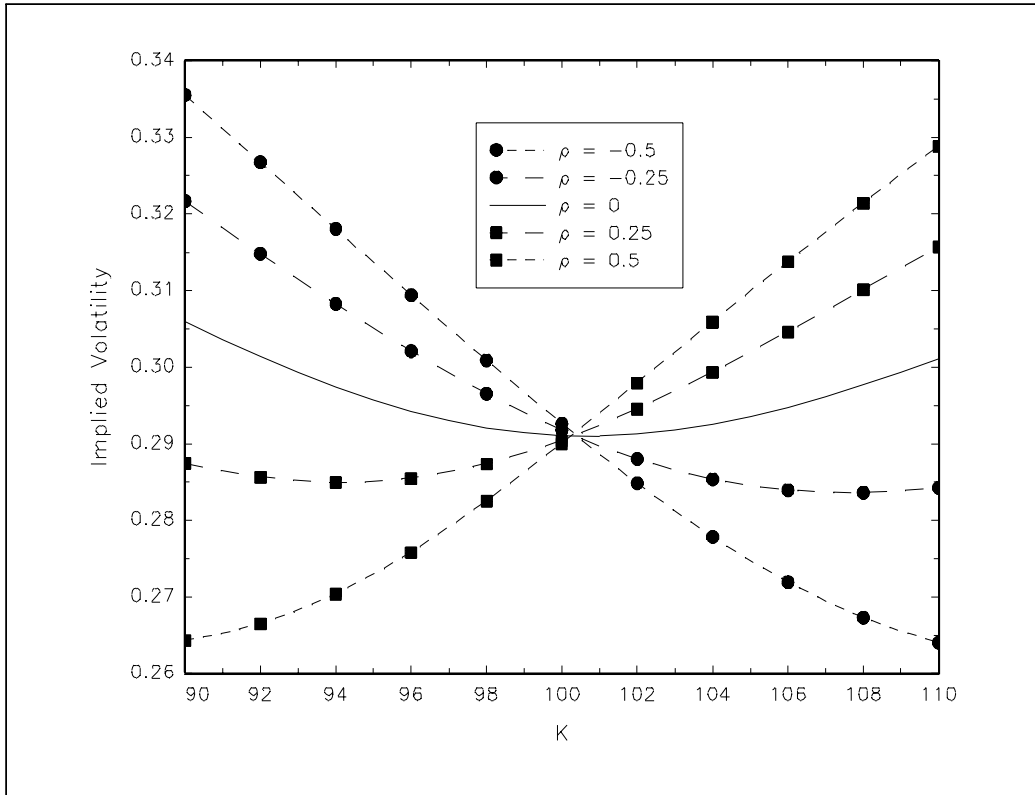
Figure 4: Smile curve : influence of ρ

depends on the calendar time, the time to maturity and the moneyness[2] of the option. The smile curve represents Black and Scholes implied volatilities across different strike prices. Figure 3 displays the volatility smile for call options on the CAC 40 quoted on October 2, 1998 in MONEP.

In this paper we assume that the observed option prices are given by the stochastic volatility option pricing model as defined in Section 1. This gives us a precise definition of the Black and Scholes implied volatility $\sigma^i\left(t, x\left(t\right), \sigma\left(t\right)\right)$ as the unique solution to

$$P\left(t, x\left(t\right), \sigma\left(t\right)\right) = P^{BS}\left(t, x\left(t\right), \sigma^i\left(t, x\left(t\right), \sigma\left(t\right)\right)\right)$$

Figure 4 represents the impact of ρ on the shape of the smile curve. For $\rho \neq 0$ we observe the skewness of the smile, which is more often encountered for options written on stocks than for interest rate or exchange rate options. The leverage effect $\rho < 0$ implies a decreasing smile curve. Clearly, the CAC 40 smile corresponds to this case.

4   Option hedging problems

Option price depends on several parameters, such as the underlying stock price and its volatility, the risk-free interest rate, and the time to maturity. The sensitivities of option prices to these arguments play a crucial role in trading and managing portfolios of options.

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[2] Renault and Touzi [1996] define moneyness of an option on the asset S with strike price K and maturity τ as

$$x\left(t\right) = \ln\left(\frac{S\left(t\right)}{K \int_t^{t+\tau} \exp\left(-r\left(u\right)\right) du}\right)$$

Practitioners consider the following option sensitivities (or greeks) :

$$\Delta = \frac{\partial P}{\partial S}$$

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 P}{\partial S^2}$$

$$\mathcal{V} = \frac{\partial P}{\partial \sigma}$$

$$R = \frac{\partial P}{\partial r}$$

$$\Theta = \frac{\partial P}{\partial t}$$

For the Black and Scholes formula, these option sensitivities can be calculated explicitly. Delta is the rate of change of the portfolio value with respect to the asset price. Gamma is the rate of change of delta with respect to the asset price. Vega is the rate of change of the portfolio value with respect to the asset's volatility.

The delta, gamma and vega measures are the most important in hedging the exposure of a portfolio of options to the market risk. The practitioners use them to quantify the different aspects of the risk inherent in their option portfolios. They attempt to make the portfolio immune to small changes in the price of the underlying asset (delta/gamma hedging) and its volatility (sigma hedging). Consequently, there is a need for very accurate computing of greeks.

Option traders find that hedging ratios computed with the Black and Scholes model and with the close-to-close historic volatility fail to achieve a well-hedged position. The usual practice to improve the hedging properties of the Black and Scholes model is to use the Black and Scholes implied volatility. However, in presence of stochastic volatility, the use of Black and Scholes implied volatilities in conjunction with the Black and Scholes computed greeks may produce various biases in option hedging strategies.

4.1 Hedging ratios biases

Following Renault and Touzi [1996], we define the *delta hedging bias* as the difference between the Black and Scholes implied volatility-based delta and the stochastic volatility model's one

$$\Delta_t^{BS}\left(x, \sigma^i\right) - \Delta_t^{SV}\left(x, \sigma\right)$$

Renault and Touzi prove that, provided we have $\rho = 0$, we verify $\forall x \geq 0$ and $\sigma > 0$

$$
\begin{aligned}
\Delta^{BS}\left(x, \sigma^i\left(x, \sigma\right)\right) &\leq \Delta^{SV}\left(x, \sigma\right) \\
\Delta^{BS}\left(-x, \sigma^i\left(-x, \sigma\right)\right) &\geq \Delta^{SV}\left(-x, \sigma\right) \\
\Delta^{BS}\left(0, \sigma^i\left(0, \sigma\right)\right) &= \Delta^{SV}\left(0, \sigma\right)
\end{aligned}
$$

For an in-the-money (out-of-the-money) option, the use of Black and Scholes implied volatility leads to an underhedged (overhedged) position.

We illustrate this in Figure 5. What is interesting, is the impact of ρ on the delta hedging bias.

We find one-signed biases when absolute values of ρ tend to one ($|\rho| \geq 0.5$ for our parameter values).

Proposition 1 $\forall x$ and $\forall \sigma > 0$ we have

$$
\begin{aligned}
\rho \to -1 &\implies \Delta^{BS}\left(x, \sigma^i\left(x, \sigma\right)\right) \leq \Delta^{SV}\left(x, \sigma\right) \\
\rho \to +1 &\implies \Delta^{BS}\left(x, \sigma^i\left(x, \sigma\right)\right) \geq \Delta^{SV}\left(x, \sigma\right)
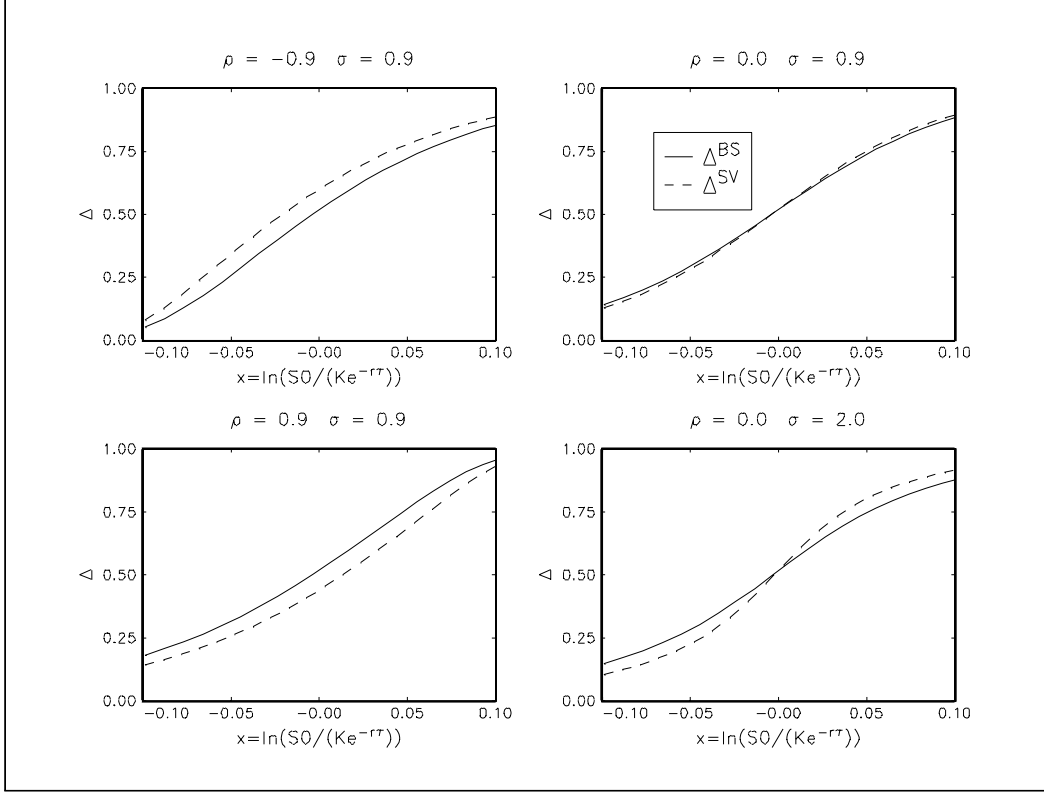\end{aligned}
$$

Figure 5: Black and Scholes implied volatility-based and the stochastic volatility model's deltas for a European call option

For ρ strongly negative, the Black and Scholes implied volatility-based delta hedging leads systematically to an underhedged position, whatever the moneyness of the option. For ρ strongly positive, the use of Black and Scholes implicit volatility leads systematically to an overhedged position.

Gamma measures the speed of option price changes in reaction to the underlying asset price modification. Therefore, gamma reflects the need of relatively frequent adjustments in the portfolio in order to keep it delta neutral. If gamma is large, delta is highly sensitive to the price of underlying asset and good management of options portfolio requires an active delta hedging. If gamma is small, delta changes slowly and rebalancing to keep a portfolio delta neutral can be made relatively less frequently.

Figure 7 compares the Black and Scholes implied volatility-based gammas with the stochastic volatility model ones. We *tend* to have the following relations

$$
\begin{array}{llll}
\rho = 0 & \implies & \Gamma^{BS}\left(x, \sigma^i\right) & \leq & \Gamma^{SV}(x, \sigma) & \text{for } x \longrightarrow 0 \\
\rho \to -1 & \implies & \Gamma^{BS}\left(x, \sigma^i\right) & \leq & \Gamma^{SV}(x, \sigma) & \text{for } x < 0 \\
\rho \to +1 & \implies & \Gamma^{BS}\left(x, \sigma^i\right) & \leq & \Gamma^{SV}(x, \sigma) & \text{for } x > 0
\end{array}
$$

Recall the feature of the smile curve for the CAC 40 call options. It implies that Black and Scholes delta-neutral positions are in this case insufficient. Moreover, in the case of out-of-the-money options, the delta hedging needs to be more dynamic.

In order to make a portfolio gamma neutral, we need a new position in a traded option. If Γ is the gamma of the portfolio and Γ_o is the gamma of a traded option, the position in traded option that makes the portfolio gamma neutral is $-\frac{\Gamma}{\Gamma_o}$. As in the pure delta hedging case, the accurate calculation of greeks is crucial.

As in the gamma case, we can make a portfolio immune to changes in volatility of the underlying asset price by taking a new $-\frac{\mathcal{V}}{\mathcal{V}_o}$ position in a traded option, where $\mathcal{V}$ and $\mathcal{V}_o$ are respectively the vegas of the
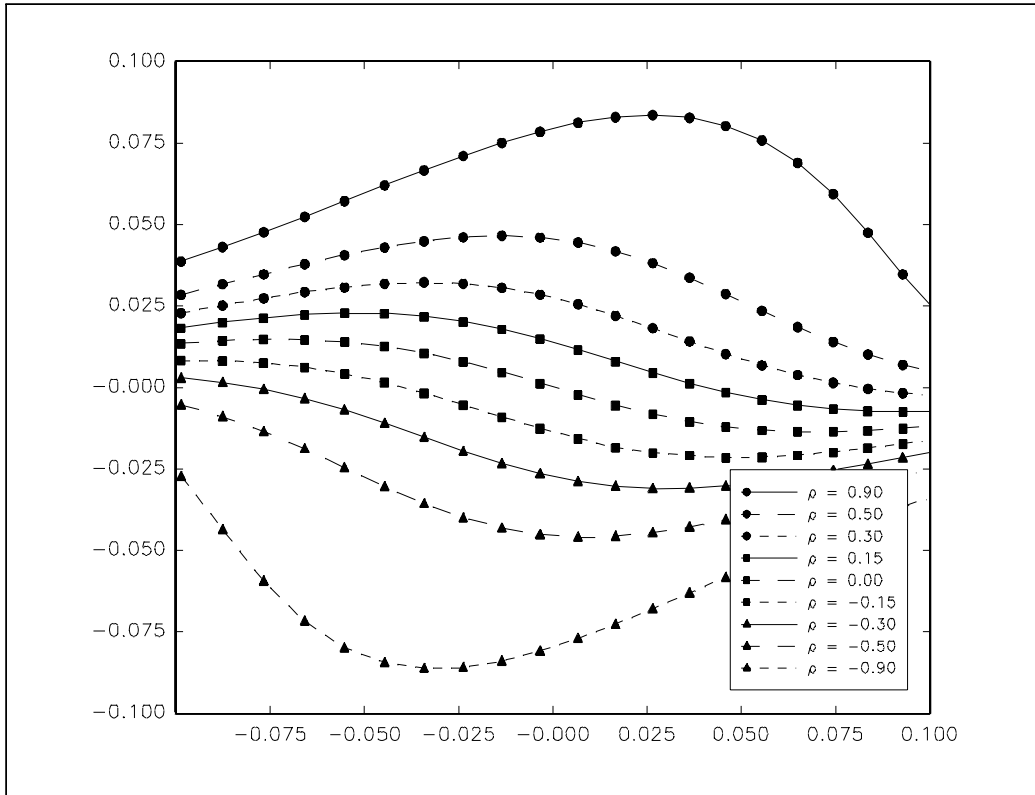
Figure 6: Delta hedging bias : influence of ρ

portfolio and of the traded option. Next, we have to readjust our position in the underlying asset in order to keep delta-neutrality of the portfolio. Such a strategy is called delta-sigma hedging. BAJEUX and ROCHET [1992] prove that in a stochastic volatility context with $\rho = 0$ the hedging problem can be solved through a delta-sigma hedging strategy : any European option completes the market.

It is shown in Figure 8 that the vega calculated from a stochastic volatility model is quite similar to the Black and Scholes vega when $\rho = 0$. It is not the case when ρ tends to 1 in absolute terms.

4.2   Option hedging strategies in stochastic volatility environment

A financial institution that sells an option faces the problem of managing its market risk. For example, the hedging problem for a financial institution that writes at time t_0 a European call option of price $C(t_0)$ consists of producing a wealth of $\text{Max}[S(T) - K, 0]$ at the maturity time $T = t_0 + \tau$.

In the Black and Scholes world, where volatilities of asset prices are constant, pure delta hedging suffices to solve the hedging problem. A short position in an option is hedged with a time-varying long position in the underlying stock. At any given time, the long positions are readjusted to equal the delta of the option position. When the hedge is rebalanced continuously, the actualized cost of this strategy is exactly equal to the price $C(t_0)$ of the option : the net hedge cost is zero. Figure 9 shows the distributions of Black and Scholes net hedge costs for weekly, daily and twice a day rebalancing. All have a mean not significantly different from zero. The standard deviations of net costs are respectively 1.01, 0.53 and 0.38, and the hedge performances[3] 0.27, 0.14 and 0.10.

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[3]Hedge performance is defined as the ratio of standard deviation of the cost of writing option and hedging it to the initial *theoretical* price of the option. HULL and WHITE [1987 b] interpret so defined hedge performance as a mesure of the percentage overpricing of the option that is necessary to provide a certain level of protection against lost.
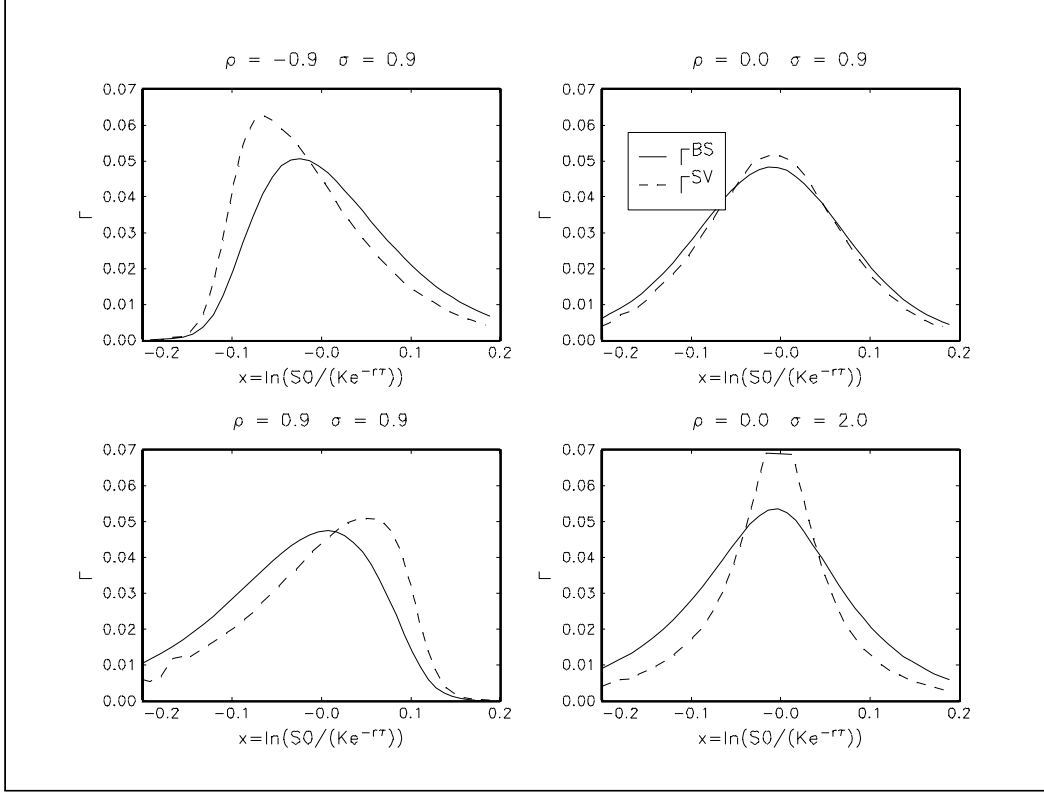
Figure 7: Patterns for variation of gamma : Black and Scholes and stochastic volatility model case

The delta-gamma strategy enables the performance of a discrete rebalanced hedging to be improved. The delta-gamma hedging scheme applied to the example of Figure 9 reduces the standard deviations of the net hedge costs to 0.28, 0.10 and 0.06 respectively for the weekly, daily and two-times a day rebalancing. The respective hedge performances become 0.08, 0.03 and 0.02.

In this section, we study the performances of hedging schemes in a stochastic volatility word as defined in section 2. More precisely, we compare performances of Black and Scholes implied volatility-based hedging with the stochastic volatility model based ones. We assume that the contingent claim prices satisfy the partial differential equation (2) with for simplicity $\lambda_2(t) = 0$ and $d = 0$.

The methodology we apply is inspired by the HULL and WHITE [1997 b] paper. Hull and White study different hedging schemes for currency options in stochastic volatility environment. However, in their calculus, they use the Black and Scholes model based formulas for greeks and option price.

4.2.1 Pure delta hedging

Consider a continuously rebalanced hedge portfolio consisting of a short position in one European call option and a long position in α underlying stocks

$$\mathcal{P}(t) = -C(t) + \alpha S(t) \tag{7}$$

The setting up of this portfolio is financed by a loan at the constant risk-free interest rate r. The instantaneous change in the value of the hedge portfolio $\mathcal{P}$ is given by

$$R(\mathcal{P}(t)) = -r\mathcal{P}(t)\,dt + d\mathcal{P}(t) \tag{8}$$

with

$$
\begin{aligned}
d\mathcal{P}(t) &= \mathcal{DP}dt + \mathcal{P}_S(t, S(t), \sigma(t))\,\sigma(t)\,S(t)\,dW_1(t) \\
&\quad + \mathcal{P}_\sigma(t, S(t), \sigma(t))\,\sigma_{2,2}(t, S(t), \sigma(t))\,dW_2(t)
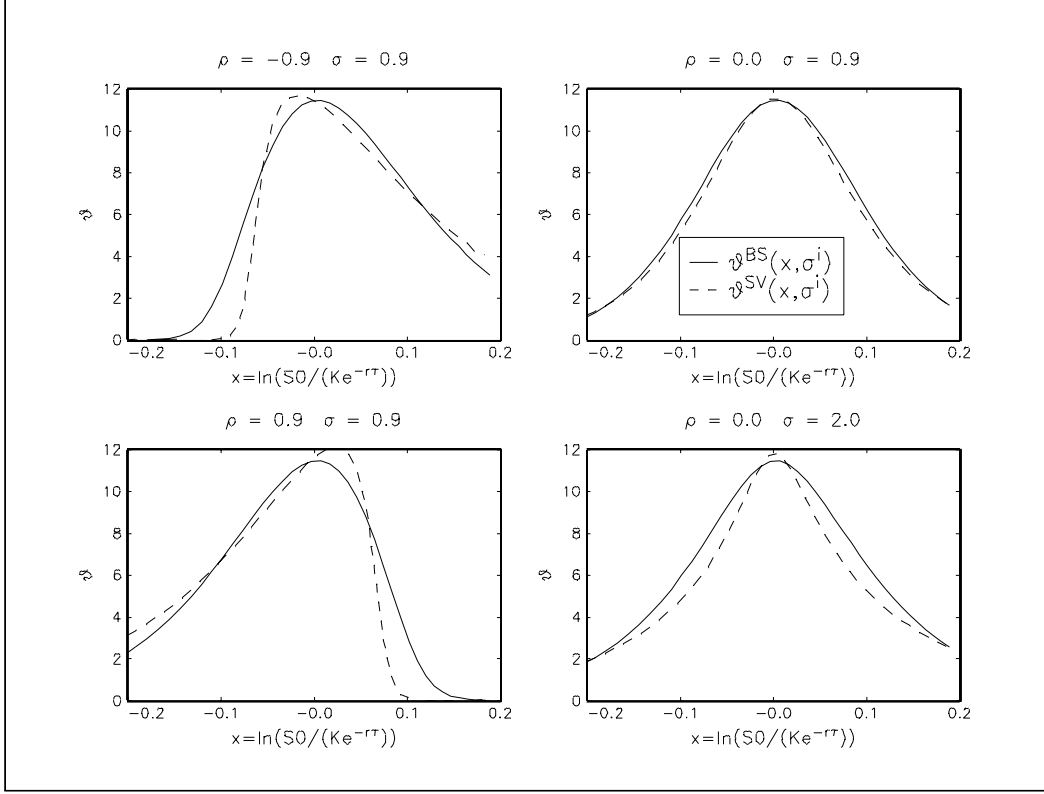\end{aligned} \tag{9}
$$

Figure 8: Patterns for variation of vega : Black and Scholes and stochastic volatility model case

where $\mathcal{D}$ is the Dynkin operator. We have with simplified notation

$$\mathcal{D}\mathcal{P} = \mathcal{P}_t + \mu S\left(t\right)\mathcal{P}_S + \mu_2\mathcal{P}_\sigma + \frac{1}{2}\sigma^2\left(t\right)S^2\left(t\right)\mathcal{P}_{SS} + \frac{1}{2}\sigma_{2,2}^2\mathcal{P}_{\sigma\sigma} + \rho\sigma\left(t\right)S\left(t\right)\sigma_{2,2}\mathcal{P}_{S\sigma}$$

The portfolio $\mathcal{P}$ is delta neutral at time t if [4]

$$\alpha = C_S = \frac{\partial C}{\partial S}\left(t, S\left(t\right), \sigma\left(t\right)\right) \tag{10}$$

Using the Black and Scholes model and the Black and Scholes implied volatility to estimate α leads to

$$\alpha^{BS} = \Delta^{BS} = \Delta^{BS}\left(t, x\left(t\right), \sigma^i\left(t, x\left(t\right), \sigma\left(t\right)\right)\right) \tag{11}$$

In this case we obtain

$$\begin{aligned}
d\mathcal{P}^{BS}\left(t\right) &= \left[-\mathcal{D}C\left(t\right) + \Delta^{BS}\mu S\left(t\right)\right]dt \\
&\quad + \left[\Delta^{BS} - C_S\right]\sigma\left(t\right)S\left(t\right)dW_1\left(t\right) + C_\sigma\sigma_{2,2}dW_2\left(t\right)
\end{aligned} \tag{12}$$

In a stochastic volatility world $C_S = \Delta^{SV}$. Noting that HB$= \Delta^{BS} - \Delta^{SV}$ gives

$$\begin{aligned}
d\mathcal{P}^{BS}\left(t\right) &= \left[-\mathcal{D}C\left(t\right) + \Delta^{BS}\mu S\left(t\right)\right]dt \\
&\quad + \text{HB}\sigma\left(t\right)S\left(t\right)dW_1\left(t\right) + C_\sigma\sigma_{2,2}dW_2\left(t\right)
\end{aligned} \tag{13}$$

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[4] When the portfolio consists of a short position in one European call option and a position in an European exchange-traded option on the same stock, we have

$$\mathcal{P}\left(t\right) = -C^1\left(t\right) + \alpha C^2\left(t\right)$$

This portfolio is delta neutral at time t if

$$\alpha = \frac{C_S^1}{C_S^2} = \frac{\frac{\partial C^1}{\partial S}\left(t, S\left(t\right), \sigma\left(t\right)\right)}{\frac{\partial C^2}{\partial S}\left(t, S\left(t\right), \sigma\left(t\right)\right)}$$
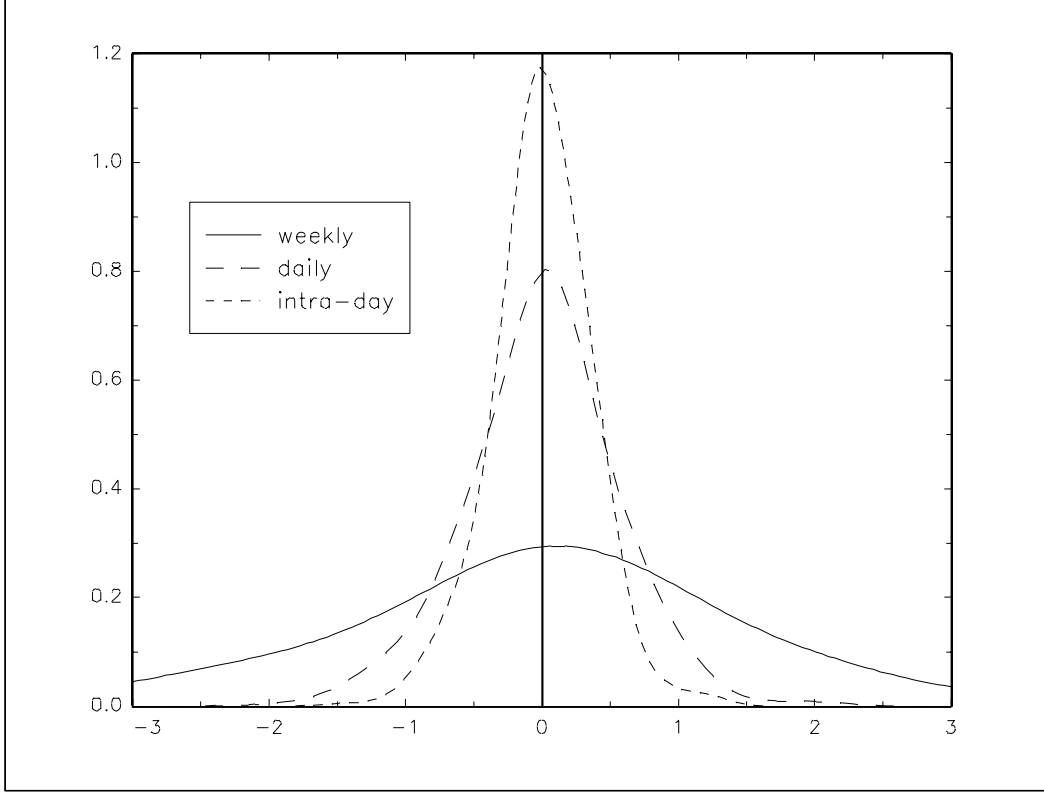
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Figure 9: Delta neutral Black and Scholes net hedge cost distributions with discrete rebalancing, $\sigma = 0.3$, $r = 0.08$ and $\tau = \frac{1}{12}$

The instantaneous change in value of the Black and Scholes implied volatility-based hedge portfolio has two stochastic components. The first arises from the delta hedging bias. The second arises from the fact that the volatility is not hedged at all. The instantaneous variance of $d\mathcal{P}^{BS}$ is

$$\frac{\text{var}\left[d\mathcal{P}^{BS}(t) \mid \mathcal{F}_t\right]}{dt} = \text{HB}^2\sigma^2(t)S^2(t) + C_\sigma^2\sigma_{2,2}^2 + 2\rho\text{HB}\sigma(t)S(t)C_\sigma\sigma_{2,2} \tag{14}$$

Now consider the delta neutral hedge portfolio based on the stochastic volatility option pricing model. In this case, we have

$$\alpha^{SV} = \Delta^{SV} = C_S(t, S(t), \sigma(t)) \tag{15}$$

and the instantaneous variance of the change in value of the hedge portfolio is

$$\frac{\text{var}\left[d\mathcal{P}^{SV}(t) \mid \mathcal{F}_t\right]}{dt} = C_\sigma^2\sigma_{2,2}^2 \tag{16}$$

In order to compare equation (14) with equation (16) suppose first that $\rho = 0$. In this case, the third term in equation (14) vanishes. Remember that for $\rho = 0$ the delta hedging bias is relatively small ($|\text{HB}| < 0.015$ for our parameter values). Consequently, the hedge performances of the Black and Scholes and stochastic volatility model based strategies may be quite similar.

When $|\rho| \to 1$, the hedging biases are greater (but always inferior to 0.1 in absolute terms for our parameter values) and the third term in equation (14) is always *negative*. The strong correlation, whatever its sign, between the asset prices' returns and their volatility tends to reduce the variance of the Black and Scholes implied volatility-based net hedge cost. In this case, the Black and Scholes based delta neutral

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hedging may outperform the delta neutral strategy based on the stochastic volatility option pricing model. In fact, when the correlation between asset prices returns and their volatility is strongly positive (negative), the Black and Scholes systematic delta overhedging (underhedging) corrects partly for the non hedge of the volatility.

We simulated the two delta neutral hedging strategies in the stochastic volatility framework of section 3 with $r = 0.08$. The maturity of the European call option was fixed to one month and the hedge was rebalanced daily. We started the simulations with $S(t_0) = K = 100$. We performed 300 replications and applied the acceleration method of antithetic variates. The table below presents the standard errors σ of the net hedge costs and the hedge performances HP that arised from our simulations.

		BS	SV
$\rho = 0$	σ	0.873	0.872
	HP	0.237	0.237
$\rho = -0.9$	σ	0.705	0.913
	HP	0.189	0.245
$\rho = 0.9$	σ	0.742	0.952
	HP	0.202	0.259

This simulation study confirms the mathematical results. The performances of the Black and Scholes and stochastic volatility based hedging schemes are similar for $\rho = 0$. For $|\rho| = 0.9$, the Black and Scholes hedge performances are better.

We notice that when volatilities of asset prices are stochastic, the overall performance of delta neutral hedging is much poorer than in the Black and Scholes world. The standard deviations of Black and Scholes net hedge costs are roughly 1.5 times greater in the presence of stochastic volatilities than when volatilities are constant. The Black and Scholes hedge performance with daily rebalancing in a stochastic volatility environment is comparable to its performance in a Black and Scholes word with hedge rebalanced only ones a week.

Failing to hedge the stochastic volatility induces a considerable risk on the financial institutions' hedged positions. This risk can be substantially reduced trough delta-sigma hedging.

4.2.2 Delta-sigma hedging

Consider a continuously rebalanced hedge portfolio consisting of a short position in one European call option, a position in α_2 units of the underlying asset and a position in α_1 units of any other exchange-traded option on the same asset

$$\mathcal{P}(t) = -C^1(t) + \alpha_1 C^2(t) + \alpha_2 S(t) \tag{17}$$

The setting up of this portfolio is financed by a loan at the constant risk-free interest rate r.

The portfolio $\mathcal{P}$ is delta and vega neutral if [5]

$$\begin{cases} \alpha_1 &= \dfrac{C_\sigma^1}{C_\sigma^2} = \dfrac{\frac{\partial C^1}{\partial \sigma}(t,S(t),\sigma(t))}{\frac{\partial C^2}{\partial \sigma}(t,S(t),\sigma(t))} \\[2ex] \alpha_2 &= C_S^1 - \alpha_1 C_S^2 = \dfrac{\partial C^1}{\partial S}(t,S(t),\sigma(t)) - \alpha_1 \dfrac{\partial C^2}{\partial S}(t,S(t),\sigma(t)) \end{cases} \tag{18}$$

---

[5] When the portfolio consists of a short position in one European call option and positions in two other exchange-traded European options on the same asset, we have

$$\mathcal{P}(t) = -C^1(t) + \alpha_1 C^2(t) + \alpha_2 C^3(t)$$

This portfolio is delta and vega neutral if

$$\begin{cases} \alpha_1 &= \dfrac{C_\sigma^1 C_S^3 - C_\sigma^3 C_S^1}{C_\sigma^2 C_S^3 - C_\sigma^3 C_S^2} \\[2ex] \alpha_2 &= \dfrac{1}{C_S^3}\left[C_S^1 - \alpha_2 C_S^2\right] \end{cases}$$

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Using the Black and Scholes model and the Black and Scholes implied volatility to estimate α_1 and α_2 leads to

$$
\begin{cases}
\alpha_1^{BS} & = \quad \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}} = \frac{\mathcal{V}_1^{BS}\left(t,x(t),\sigma^i(t,x(t),\sigma(t))\right)}{\mathcal{V}_2^{BS}(t,x(t),\sigma^i(t,x(t),\sigma(t)))} \\
\\
\alpha_2^{BS} & = \quad \Delta_1^{BS} - \alpha_1^{BS}\Delta_2^{BS} \\
& = \quad \Delta_1^{BS}\left(t,x\left(t\right),\sigma^i\left(t,x\left(t\right),\sigma\left(t\right)\right)\right) - \alpha_1^{BS}\Delta_2^{BS}\left(t,x\left(t\right),\sigma^i\left(t,x\left(t\right),\sigma\left(t\right)\right)\right)
\end{cases}
\tag{19}
$$

In this case we obtain

$$
\begin{aligned}
d\mathcal{P}^{BS}(t) & = \quad \left[-\mathcal{D}C^1 + \alpha_1^{BS}\mathcal{D}C^2 + \alpha_2^{BS}\mu S(t)\right]dt \\
& \quad + \left[-C_S^1 + \alpha_1^{BS}C_S^2 + \alpha_2^{BS}\right]\sigma(t)S(t)\,dW_1(t) \\
& \quad + \left[-C_\sigma^1 + \alpha_1^{BS}C_\sigma^2\right]\sigma_{2,2}\,dW_2(t)
\end{aligned}
\tag{20}
$$

In a stochastic volatility world $C_S^i = \Delta_i^{SV}$ and $C_\sigma^i = \mathcal{V}_i^{SV}$, $i = 1, 2$. Noting that $\mathrm{HB}_1 = \Delta_1^{BS} - \Delta_1^{SV}$ and $\mathrm{HB}_2 = \Delta_2^{BS} - \Delta_2^{SV}$ gives

$$
\begin{aligned}
d\mathcal{P}^{BS}(t) & = \quad \left[-\mathcal{D}C^1 + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathcal{D}C^2 + \left(\Delta_1^{BS} - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\Delta_2^{BS}\right)\mu S(t)\right]dt \\
& \quad + \left[\mathrm{HB}_1 - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathrm{HB}_2\right]\sigma(t)S(t)\,dW_1(t) \\
& \quad + \left[-\mathcal{V}_1^{SV} + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathcal{V}_2^{SV}\right]\sigma_{2,2}\,dW_2(t)
\end{aligned}
\tag{21}
$$

The instantaneous change in the value of the Black and Scholes implied volatility-based hedge portfolio has two stochastic components which arise from the delta and vega hedging biases. The instantaneous variance of $d\mathcal{P}^{BS}$ is

$$
\begin{aligned}
\frac{1}{dt}\mathrm{var}\left[d\mathcal{P}^{BS}(t) \mid \mathcal{F}_t\right] & = \quad \left[\mathrm{HB}_1 - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathrm{HB}_2\right]^2\sigma^2(t)S^2(t) + \left[-\mathcal{V}_1^{SV} + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathcal{V}_2^{SV}\right]^2\sigma_{2,2}^2 \\
& \quad + 2\rho\left[\mathrm{HB}_1 - \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathrm{HB}_2\right]\left[-\mathcal{V}_1^{SV} + \frac{\mathcal{V}_1^{BS}}{\mathcal{V}_2^{BS}}\mathcal{V}_2^{SV}\right]\sigma(t)S(t)\sigma_{2,2}
\end{aligned}
\tag{22}
$$

Now consider the delta-sigma hedging portfolio based on the stochastic volatility option pricing model. In this case, we have

$$
\begin{cases}
\alpha_1^{SV} & = \quad \frac{\mathcal{V}_1^{SV}}{\mathcal{V}_2^{SV}} = \frac{C_\sigma^1(t,S(t),\sigma(t))}{C_\sigma^2(t,S(t),\sigma(t))} \\
\\
\alpha_2^{SV} & = \quad \Delta_1^{SV} - \alpha_1^{SV}\Delta_2^{SV} = C_S^1\left(t,S\left(t\right),\sigma\left(t\right)\right) - \alpha_1^{SV}C_S^2\left(t,S\left(t\right),\sigma\left(t\right)\right)
\end{cases}
\tag{23}
$$

If the hedge is rebalanced continuously, the instantaneous variance in the value of this portfolio is zero : the stochastic volatility model based delta-sigma scheme solves the hedging problem.

Suppose that $\rho = 0$. In this case, the third term in equation (22) vanishes. Moreover, the delta and vega hedging biases are relatively small. Consequently, the Black and Scholes implied volatility-based delta-sigma hedging performs almost as well as the delta-sigma hedging based on the stochastic volatility model.

When the correlation between asset prices' returns and their volatility is strong in absolute terms, the Black and Scholes systematic one-signed delta hedging biases on the two options C_1 and C_2 tend to compensate. This effect reduces the instantaneous variance of the Black and Scholes delta-sigma hedge portfolio.

We undertook a similar Monte Carlo study to that in sub-section 4.2.1. The hedged option has a one month maturity, $\tau_1 = \frac{30}{360}$. The maturity of the second exchange-traded option was fixed at six months, $\tau_2 = \frac{1}{2}$. The two options start at-the-money. The hedge was rebalanced daily. We obtain the following hedge performances HP and standard errors σ of the net hedge costs for the two delta-sigma hedging schemes.

		BS	SV
$\rho = 0$	σ	0.497	0.488
	HP	0.135	0.133
$\rho = -0.9$	σ	0.497	0.491
	HP	0.134	0.132
$\rho = 0.9$	σ	0.486	0.486
	HP	0.132	0.132

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We can see clearly that the hedge performances in a stochastic volatility environment are substantially improved when one uses the delta-sigma hedging scheme. The variability of hedge positions is reduced significantly (by roughly 40%) with respect to the pure delta neutral hedging. The standard deviations of the net hedge costs and the delta-sigma hedge performances are comparable to those obtained for the delta neutral hedging in the Black and Scholes world.

We notice that, although the stochastic volatility model based delta-sigma hedge tends to perform slightly better, the differences with the Black and Scholes implied volatility-based strategy are not significant. In fact, as suggested by Hull and White [1987 b], it is important to distinguish between the effect of stochastic volatility on the value of option and on its partial derivatives *at one point in time* and the effect of stochastic volatility on the performance *over time* of different hedging schemes. The first effect (hedging ratios biases) may be significant or not, depending on the option's moneyness and maturity and on parameter values of the underlying stochastic volatility data generating process. However, errors in greeks calculus may swamp mutually in a dynamic setting. The Black and Scholes implied volatility-based hedging, although logically inconsistent, may in this case perform just as well as the stochastic volatility model based hedging strategies. What matters, is not to neglect the risk that the stochastic behavior of volatilities induces in option positions. Delta-sigma hedging enables it to be reduced considerably.

4.2.3 Correcting for discrete hedge rebalancing

The use of second partial derivatives of the option price with respect to the state variables enables the performance of discretely rebalanced hedging to be improved. Gamma is the rate of change of delta with respect to the underlying asset price. It is relatively large for the near-the-money options. The delta-gamma strategy is a well known hedging scheme. On the other hand, the rate of change of vega with respect to the volatility of the underlying asset price is not a commonly used hedging ratio. Vega seems to be sensitive to the volatility of the underlying asset price only for the relatively deep out-of-the-money and in-the-money options. For the options near-the-money, the sensitivity of vega tends to zero.

The **delta-gamma** scheme consists in hedging a short position in an option with a position in α_2 units of the underlying asset and a position in α_1 units of any other exchange traded option on the same asset

$$\mathcal{P}(t) = -C^1(t) + \alpha_1(t) C^2(t) + \alpha_2(t) S(t) \tag{24}$$

The setting up of the portfolio $\mathcal{P}$ is financed by a loan at the constant risk-free interest rate. The hedge is not rebalanced between times t and $t + \Delta t$. We have

$$\mathcal{P}(t + \Delta t) = -C^1(t + \Delta t) + \alpha_1(t) C^2(t + \Delta t) + \alpha_2(t) S(t + \Delta t) \tag{25}$$

and

$$\Delta \mathcal{P}(t) = -\Delta C^1(t) + \alpha_1 \Delta C^2(t) + \alpha_2 \Delta S(t) \tag{26}$$

Following Hull and White [1987 b], we can expand $\Delta \mathcal{P}(t)$ in Taylor series about $S(t)$, $\sigma(t)$ and t to obtain, with simplified notations

$$\begin{aligned} \Delta \mathcal{P} = & \left[-C_S^1 + \alpha_1 C_S^2 + \alpha_2 \right] \Delta S + \left[-C_\sigma^1 + \alpha_1 C_\sigma^2 \right] \Delta \sigma + \left[-C_t^1 + \alpha_1 C_t^2 \right] \Delta t \\ & + \frac{1}{2} \left[-C_{SS}^1 + \alpha_1 C_{SS}^2 \right] (\Delta S)^2 + \frac{1}{2} \left[-C_{\sigma\sigma}^1 + \alpha_1 C_{\sigma\sigma}^2 \right] (\Delta \sigma)^2 + \frac{1}{2} \left[-C_{tt}^1 + \alpha_1 C_{tt}^2 \right] (\Delta t)^2 \\ & + \left[-C_{S\sigma}^1 + \alpha_1 C_{S\sigma}^2 \right] \Delta S \Delta \sigma + \left[-C_{St}^1 + \alpha_1 C_{St}^2 \right] \Delta S \Delta t + \left[-C_{\sigma t}^1 + \alpha_1 C_{\sigma t}^2 \right] \Delta \sigma \Delta t + \dots \end{aligned} \tag{27}$$

The portfolio $\mathcal{P}$ is delta and gamma neutral at time t if

$$\begin{cases} \alpha_1 = \frac{C_{SS}^1}{C_{SS}^2} = \frac{\frac{\partial^2 C^1}{\partial S^2}(t, S(t), \sigma(t))}{\frac{\partial^2 C^2}{\partial S^2}(t, S(t), \sigma(t))} \\ \\ \alpha_2 = C_S^1 - \alpha_1(t) C_S^2 = \frac{\partial C^1}{\partial S}(t, S(t), \sigma(t)) - \alpha_1 \frac{\partial C^2}{\partial S}(t, S(t), \sigma(t)) \end{cases} \tag{28}$$

Using the Black and Scholes model and the Black and Scholes implied volatility to estimate α_1 and α_2 leads to

$$\begin{cases} \alpha_1^{BS} = \frac{\Gamma_1^{BS}}{\Gamma_2^{BS}} = \frac{\Gamma_1^{BS}\left(t, x(t), \sigma^i(t, x(t), \sigma(t))\right)}{\Gamma_2^{BS}\left(t, x(t), \sigma^i(t, x(t), \sigma(t))\right)} \\ \\ \alpha_2^{BS} = \Delta_1^{BS} - \alpha_1^{BS} \Delta_2^{BS} \\ \quad\;\; = \Delta_1^{BS}\left(t, x(t), \sigma^i(t, x(t), \sigma(t))\right) - \alpha_1 \Delta_2^{BS}\left(t, x(t), \sigma^i(t, x(t), \sigma(t))\right) \end{cases} \tag{29}$$

Noting that in the stochastic volatility environment $C_S^i = \Delta_i^{SV}$, $C_{SS}^i = \Gamma_i^{SV}$ and $C_\sigma^i = \mathcal{V}_i^{SV}$, $i = 1, 2$ gives

$$
\begin{aligned}
\Delta\mathcal{P}^{BS} &= \left[\text{HB}_1 - \alpha_1^{BS}\text{HB}_2\right]\Delta S + \left[-\mathcal{V}_1^{SV} + \alpha_1^{BS}\mathcal{V}_2^{SV}\right]\Delta\sigma + \left[-C_t^1 + \alpha_1^{BS}C_t^2\right]\Delta t \\
&+ \tfrac{1}{2}\left[-\Gamma_1^{SV} + \alpha_1^{BS}\Gamma_2^{SV}\right](\Delta S)^2 + \tfrac{1}{2}\left[-C_{\sigma\sigma}^1 + \alpha_1^{BS}C_{\sigma\sigma}^2\right](\Delta\sigma)^2 + \tfrac{1}{2}\left[-C_{tt}^1 + \alpha_1^{BS}C_{tt}^2\right](\Delta t)^2 \\
&+ \left[-C_{S\sigma}^1 + \alpha_1^{BS}C_{S\sigma}^2\right]\Delta S\Delta\sigma + \left[-C_{St}^1 + \alpha_1^{BS}C_{St}^2\right]\Delta S\Delta t + \left[-C_{\sigma t}^1 + \alpha_1^{BS}C_{\sigma t}^2\right]\Delta\sigma\Delta t + \cdots
\end{aligned}
\tag{30}
$$

The variance of $\Delta\mathcal{P}^{BS}$ can be approximated as

$$
\begin{aligned}
\text{var}\left[\Delta\mathcal{P}^{BS} \mid \mathcal{F}_t\right] &\simeq \left[-\mathcal{V}_1^{SV} + \alpha_1^{BS}\mathcal{V}_2^{SV}\right]^2 \text{var}\left[\Delta\sigma \mid \mathcal{F}_t\right] + \tfrac{1}{4}\left[-C_{\sigma\sigma}^1 + \alpha_1^{BS}C_{\sigma\sigma}^2\right]^2 \text{var}\left[(\Delta\sigma)^2 \mid \mathcal{F}_t\right] \\
&+ \left[\text{HB}_1 - \alpha_1^{BS}\text{HB}_2\right]^2 \text{var}\left[\Delta S \mid \mathcal{F}_t\right] + \tfrac{1}{4}\left[-\Gamma_1^{SV} + \alpha_1^{BS}\Gamma_2^{SV}\right]^2 \text{var}\left[(\Delta S)^2 \mid \mathcal{F}_t\right] \\
&+ 2\left[\text{HB}_1 - \alpha_1^{BS}\text{HB}_2\right]\left[-\mathcal{V}_1^{SV} + \alpha_1^{BS}\mathcal{V}_2^{SV}\right]\text{cov}\left[\Delta S, \Delta\sigma \mid \mathcal{F}_t\right] \\
&+ \left[-\mathcal{V}_1^{SV} + \alpha_1^{BS}\mathcal{V}_2^{SV}\right]\left[-\Gamma_1^{SV} + \alpha_1^{BS}\Gamma_2^{SV}\right]\text{cov}\left[(\Delta S)^2, \Delta\sigma \mid \mathcal{F}_t\right] \\
&+ \left[\text{HB}_1 - \alpha_1^{BS}\text{HB}_2\right]\left[-C_{\sigma\sigma}^1 + \alpha_1^{BS}C_{\sigma\sigma}^2\right]\text{cov}\left[\Delta S, (\Delta\sigma)^2 \mid \mathcal{F}_t\right]
\end{aligned}
\tag{31}
$$

Now consider the delta-gamma hedging portfolio based on the stochastic volatility option pricing model. In this case we have

$$
\begin{cases}
\alpha_1^{SV} &= \dfrac{\Gamma_1^{SV}}{\Gamma_2^{SV}} = \dfrac{C_{SS}^1(t, S(t), \sigma(t))}{C_{SS}^2(t, S(t), \sigma(t))} \\[2ex]
\alpha_2^{SV} &= \Delta_1^{SV} - \alpha_1^{SV}\Delta_2^{SV} = C_S^1(t, S(t), \sigma(t)) - \alpha_1^{SV}C_S^2(t, S(t), \sigma(t))
\end{cases}
\tag{32}
$$

and the variance of $\Delta\mathcal{P}^{SV}$ is approximated by

$$
\begin{aligned}
\text{var}\left[\Delta\mathcal{P}^{SV} \mid \mathcal{F}_t\right] &\simeq \left[-\mathcal{V}_1^{SV} + \alpha_1^{SV}\mathcal{V}_2^{SV}\right]^2 \text{var}\left[\Delta\sigma \mid \mathcal{F}_t\right] + \tfrac{1}{4}\left[-C_{\sigma\sigma}^1 + \alpha_1^{SV}C_{\sigma\sigma}^2\right]^2 \text{var}\left[(\Delta\sigma)^2 \mid \mathcal{F}_t\right] \\
&\simeq \left[-\mathcal{V}_1^{SV} + \dfrac{\Gamma_1^{SV}}{\Gamma_2^{SV}}\mathcal{V}_2^{SV}\right]^2 \sigma_{2,2}^2 \Delta t + \tfrac{1}{2}\left[-C_{\sigma\sigma}^1 + \alpha_1^{SV}C_{\sigma\sigma}^2\right]^2 \sigma_{2,2}^4 (\Delta t)^2
\end{aligned}
\tag{33}
$$

It is difficult to compare analytically the performance of these two delta-gamma strategies. The table below presents the results of our Monte Carlo study. The hedged option has a one month maturity. The maturity of the second exchange-traded option was fixed to six months in the first instance and to 45 days in the second. The two options start at-the-money. The hedge was rebalanced daily.

		BS		SV	
		$\tau_2 = \frac{1}{2}$	$\tau_2 = \frac{45}{360}$	$\tau_2 = \frac{1}{2}$	$\tau_2 = \frac{45}{360}$
$\rho = 0$	σ	1.165	0.763	1.184	0.808
	HP	0.316	0.207*	0.321	0.219*
$\rho = -0.9$	σ	0.895	0.713	1.825	1.188
	HP	0.241	0.192	0.490	0.319
$\rho = 0.9$	σ	1.479	0.906	1.014	0.555
	HP	0.403	0.247	0.276	0.151*

The asterisk means a higher performance with respect to the pure delta neutral strategy.

We remark that in a stochastic volatility environment the hedge performance of delta-gamma strategies deteriorates with the maturity of the exchange-traded option. For τ_2 high, the effect of stochastic volatility swamps the advantage of correcting for discrete hedge rebalancing. Consider equation (33) and neglect its second term. For $\tau_2 > \tau_1 \gg 0$, Γ_2^{SV} is a decreasing function of τ_2 and $\mathcal{V}_2^{SV}$ is an increasing function of τ_2. Moreover, $\frac{\Gamma_1^{SV}}{\Gamma_2^{SV}} > 1$ and $\mathcal{V}_2^{SV} > \mathcal{V}_1^{SV}$. In this case, $\text{var}\left[\Delta\mathcal{P}^{SV} \mid \mathcal{F}_t\right]$ becomes larger when τ_2 increases. The same is true for $\text{var}\left[\Delta\mathcal{P}^{BS} \mid \mathcal{F}_t\right]$. This result confirms the conclusion of Hull and White [1987 b].

For $\tau_2 = \frac{1}{2}$, the delta-gamma hedge performance is always poorer than the performance of pure delta neutral hedging. For $\tau_2 = \frac{45}{360}$, the relative performance of the delta-gamma and the pure delta hedging strategies depends on ρ and the model used for the calculus of hedging ratios.

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When $\rho \leq 0$, the Black and Scholes implied volatility-based delta-gamma hedging performs better than the delta-gamma scheme based on the stochastic volatility model. For $\rho > 0$ the inverse is true.

We remark that when volatilities of asset prices are stochastic, the overall performance of delta-gamma hedging is significantly poorer than the performance of the delta-sigma strategies. As we have seen in section 4.2.1, failing to hedge the stochastic volatility is significantly risky.

The **delta-gamma-sigma** scheme consists in hedging a short position in an option with a position in α_3 units of the underlying asset and two positions in respectively α_1 and α_2 units of any other exchange traded options on the same asset

$$\mathcal{P}(t) = -C^1(t) + \alpha_1(t) C^2(t) + \alpha_2(t) C^3(t) + \alpha_3(t) S(t) \tag{34}$$

The setting up of the portfolio $\mathcal{P}$ is financed by a loan at the risk-free interest rate. The hedge is not rebalanced between times t and $t + \Delta t$. Applying the same methodology as above, we obtain

$$
\begin{aligned}
\Delta \mathcal{P} =\ & \left[-C_S^1 + \alpha_1 C_S^2 + \alpha_2 C_S^3 \right] \Delta S + \left[-C_\sigma^1 + \alpha_1 C_\sigma^2 + \alpha_2 C_\sigma^3 \right] \Delta \sigma \\
& + \left[-C_t^1 + \alpha_1 C_t^2 + \alpha_2 C_t^3 \right] \Delta t + \tfrac{1}{2} \left[-C_{tt}^1 + \alpha_1 C_{tt}^2 + \alpha_2 C_{tt}^3 \right] (\Delta t)^2 \\
& + \tfrac{1}{2} \left[-C_{SS}^1 + \alpha_1 C_{SS}^2 + \alpha_2 C_{SS}^3 \right] (\Delta S)^2 + \tfrac{1}{2} \left[-C_{\sigma\sigma}^1 + \alpha_1 C_{\sigma\sigma}^2 + \alpha_2 C_{\sigma\sigma}^3 \right] (\Delta \sigma)^2 \\
& + \left[-C_{S\sigma}^1 + \alpha_1 C_{S\sigma}^2 + \alpha_2 C_{S\sigma}^3 \right] \Delta S \Delta \sigma + \left[-C_{St}^1 + \alpha_1 C_{St}^2 + \alpha_2 C_{St}^3 \right] \Delta S \Delta t \\
& + \left[-C_{\sigma t}^1 + \alpha_1 C_{\sigma t}^2 + \alpha_2 C_{\sigma t}^3 \right] \Delta \sigma \Delta t + \dots
\end{aligned} \tag{35}
$$

The portfolio $\mathcal{P}$ is delta, gamma and vega neutral at time t if

$$
\begin{cases}
\alpha_1 &=\ \dfrac{C_{SS}^3 C_\sigma^1 - C_{SS}^1 C_\sigma^3}{C_\sigma^2 C_{SS}^3 - C_{SS}^2 C_\sigma^3} \\[2mm]
\alpha_2 &=\ \dfrac{1}{C_{SS}^3} \left(C_{SS}^1 - \alpha_1 C_{SS}^2 \right) \\[2mm]
\alpha_3 &=\ C_S^1 - \alpha_1 C_S^2 - \alpha_2 C_S^3
\end{cases} \tag{36}
$$

Using the Black and Scholes model and the Black and Scholes implied volatility to evaluate α_1, α_2 and α_3 leads to

$$
\begin{cases}
\alpha_1^{BS} &=\ \dfrac{\Gamma_3^{BS} \mathcal{V}_1^{BS} - \Gamma_1^{BS} \mathcal{V}_3^{BS}}{\Gamma_3^{BS} \mathcal{V}_2^{BS} - \Gamma_2^{BS} \mathcal{V}_3^{BS}} \\[2mm]
\alpha_2^{BS} &=\ \dfrac{1}{\Gamma_3^{BS}} \left(\Gamma_1^{BS} - \alpha_1^{BS} \Gamma_2^{BS} \right) \\[2mm]
\alpha_3^{BS} &=\ \Delta_1^{BS} - \alpha_1^{BS} \Delta_2^{BS} - \alpha_2^{BS} \Delta_3^{BS}
\end{cases} \tag{37}
$$

The change in value of the Black and Scholes delta-gamma-sigma hedge portfolio is then given by

$$
\begin{aligned}
\Delta \mathcal{P}^{BS} =\ & \left[\text{HB}_1 - \alpha_1^{BS} \text{HB}_2 - \alpha_2^{BS} \text{HB}_3 \right] \Delta S + \left[-\mathcal{V}_1^{SV} + \alpha_1^{BS} \mathcal{V}_2^{SV} + \alpha_2^{BS} \mathcal{V}_3^{SV} \right] \Delta \sigma \\
& + \left[-C_t^1 + \alpha_1^{BS} C_t^2 + \alpha_2^{BS} C_t^3 \right] \Delta t + \tfrac{1}{2} \left[-C_{tt}^1 + \alpha_1^{BS} C_{tt}^2 + \alpha_2^{BS} C_{tt}^3 \right] (\Delta t)^2 \\
& + \tfrac{1}{2} \left[-\Gamma_1^{SV} + \alpha_1^{BS} \Gamma_2^{SV} + \alpha_2^{BS} \Gamma_3^{SV} \right] (\Delta S)^2 + \tfrac{1}{2} \left[-C_{\sigma\sigma}^1 + \alpha_1^{BS} C_{\sigma\sigma}^2 + \alpha_2^{BS} C_{\sigma\sigma}^3 \right] (\Delta \sigma)^2 \\
& + \left[-C_{S\sigma}^1 + \alpha_1^{BS} C_{S\sigma}^2 + \alpha_2^{BS} C_{S\sigma}^3 \right] \Delta S \Delta \sigma + \left[-C_{St}^1 + \alpha_1^{BS} C_{St}^2 + \alpha_2^{BS} C_{St}^3 \right] \Delta S \Delta t \\
& + \left[-C_{\sigma t}^1 + \alpha_1^{BS} C_{\sigma t}^2 + \alpha_2^{BS} C_{\sigma t}^3 \right] \Delta \sigma \Delta t + \dots
\end{aligned} \tag{38}
$$

and its variance can be approximated as

$$
\begin{aligned}
\text{var} \left[\Delta \mathcal{P}^{BS} \mid \mathcal{F}_t \right] \simeq\ & \left[-\mathcal{V}_1^{SV} + \alpha_1^{BS} \mathcal{V}_2^{SV} + \alpha_2^{BS} \mathcal{V}_3^{SV} \right]^2 \text{var} \left[\Delta \sigma \mid \mathcal{F}_t \right] \\
& + \tfrac{1}{4} \left[-C_{\sigma\sigma}^1 + \alpha_1^{BS} C_{\sigma\sigma}^2 + \alpha_2^{BS} C_{\sigma\sigma}^3 \right]^2 \text{var} \left[(\Delta \sigma)^2 \mid \mathcal{F}_t \right] \\
& + \left[\text{HB}_1 - \alpha_1^{BS} \text{HB}_2 - \alpha_2^{BS} \text{HB}_3 \right]^2 \text{var} \left[\Delta S \mid \mathcal{F}_t \right] \\
& + \tfrac{1}{4} \left[-\Gamma_1^{SV} + \alpha_1^{BS} \Gamma_2^{SV} + \alpha_2^{BS} \Gamma_3^{SV} \right]^2 \text{var} \left[(\Delta S)^2 \mid \mathcal{F}_t \right] \\
& + 2 \left[\text{HB}_1 - \alpha_1^{BS} \text{HB}_2 - \alpha_2^{BS} \text{HB}_3 \right] \left[-\mathcal{V}_1^{SV} + \alpha_1^{BS} \mathcal{V}_2^{SV} + \alpha_2^{BS} \mathcal{V}_3^{SV} \right] \text{cov} \left[\Delta S, \Delta \sigma \mid \mathcal{F}_t \right] \\
& + \left[-\mathcal{V}_1^{SV} + \alpha_1^{BS} \mathcal{V}_2^{SV} + \alpha_2^{BS} \mathcal{V}_3^{SV} \right] \left[-\Gamma_1^{SV} + \alpha_1^{BS} \Gamma_2^{SV} + \alpha_2^{BS} \Gamma_3^{SV} \right] \text{cov} \left[(\Delta S)^2, \Delta \sigma \mid \mathcal{F}_t \right] \\
& + \left[\text{HB}_1 - \alpha_1^{BS} \text{HB}_2 - \alpha_2^{BS} \text{HB}_3 \right] \left[-C_{\sigma\sigma}^1 + \alpha_1^{BS} C_{\sigma\sigma}^2 + \alpha_2^{BS} C_{\sigma\sigma}^3 \right] \text{cov} \left[\Delta S, (\Delta \sigma)^2 \mid \mathcal{F}_t \right]
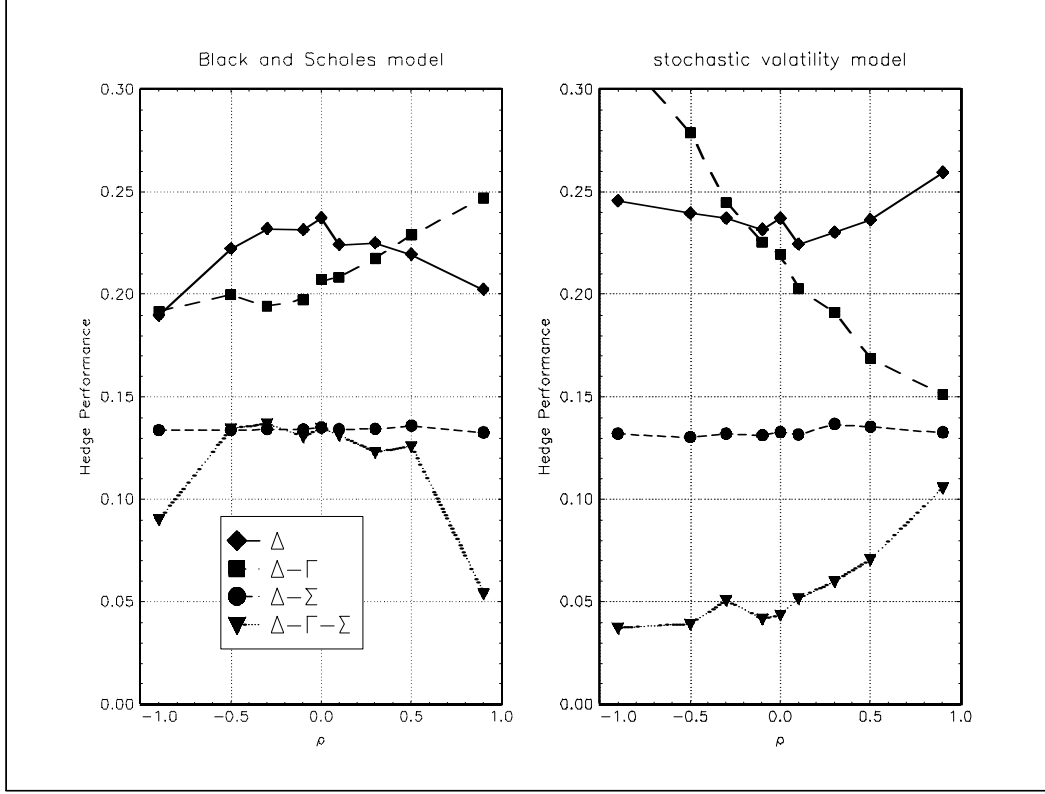\end{aligned} \tag{39}
$$

Figure 10: Hedge performances in a stochastic volatility environment for different values of ρ.

An approximation for the variance of the change in value of the delta-gamma-sigma hedge portfolio based on the stochastic volatility model for option pricing is given by

$$
\begin{aligned}
\text{var}\left[\Delta \mathcal{P}^{SV} \mid \mathcal{F}_t\right] &\simeq \tfrac{1}{4}\left[-C_{\sigma\sigma}^1 + \alpha_1^{SV}C_{\sigma\sigma}^2 + \alpha_2^{SV}C_{\sigma\sigma}^3\right]^2 \text{var}\left[(\Delta\sigma)^2 \mid \mathcal{F}_t\right] \\
&\simeq \tfrac{1}{2}\left[-C_{\sigma\sigma}^1 + \alpha_1^{SV}C_{\sigma\sigma}^2 + \alpha_2^{SV}C_{\sigma\sigma}^3\right]^2 \sigma_{2,2}^4 \left(\Delta t\right)^2
\end{aligned}
\tag{40}
$$

Monte Carlo simulations enable us to analyse the hedge performance of these two delta-gamma-sigma strategies. The hedged option has a maturity of one month. The maturities of the two exchange-traded options were fixed to three and six months, $\tau_2 = \frac{1}{4}$, $\tau_3 = \frac{1}{2}$. All the three options start at the money. The hedge was rebalanced daily.

		BS	SV
$\rho = 0$	σ	0.496	0.161
	HP	0.135	0.044
$\rho = -0.9$	σ	0.336	0.139
	HP	0.090	0.037
$\rho = 0.9$	σ	0.193	0.362
	HP	0.053	0.099

We remark that in almost all the cases the daily rebalanced delta-gamma-sigma strategies significantly outperform the delta-sigma hedging scheme. Correcting for discrete hedge rebalancing improves substantially performances of the delta-sigma strategy. The exception is the case of Black and Scholes implied volatility-based hedging for $\rho = 0$.

When $\rho \leq 0$, the hedge performance of the delta-gamma-sigma strategy based on the stochastic volatility model for option pricing is more than two times better than the hedge performance of the scheme based on
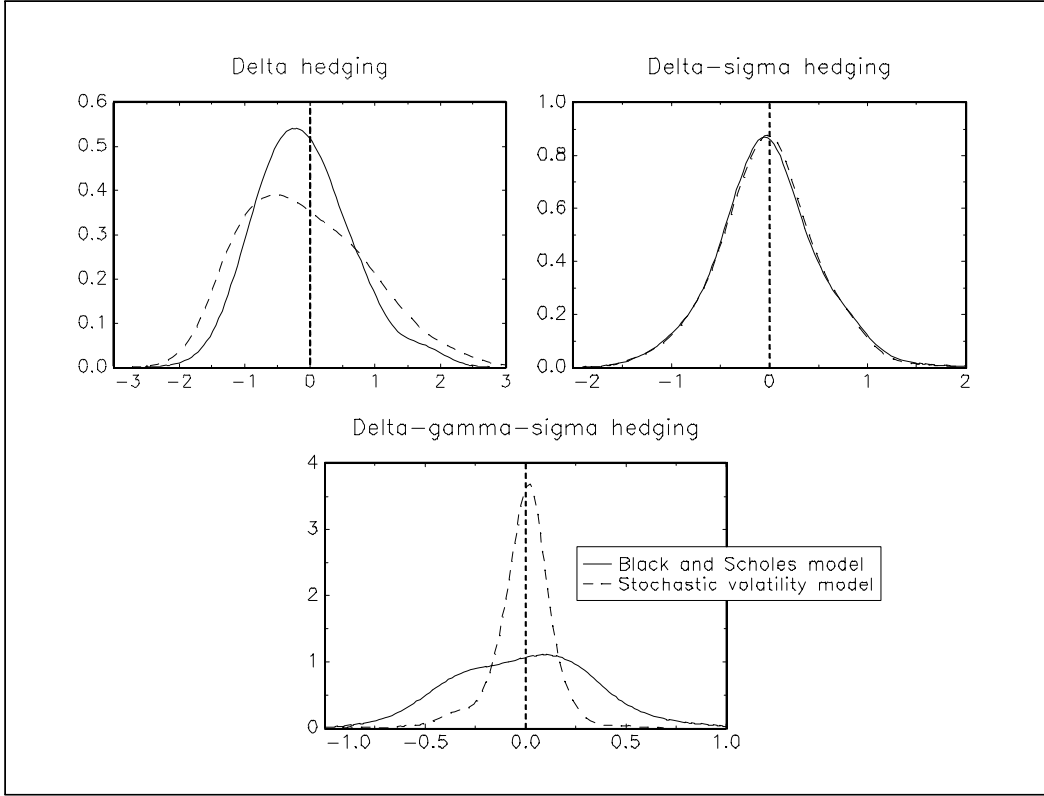
Figure 11: Black and Scholes and stochastic volatility model based net hedge cost distributions in the case of $\rho = -0.9$.

the Black and Scholes model. It is not the case for $\rho > 0$, where the Black and Scholes model outperforms the stochastic volatility methods.

When $\rho \leq 0$, the delta-gamma-sigma scheme based on stochastic volatility model reduces the variability of hedge positions by roughly 67 % with respect to the delta-sigma hedge and by roughly 82 % with respect to the pure delta hedging scheme. The hedge performances of the stochastic volatility model based delta-gamma-sigma strategy are in this case comparable to the performance of the delta-gamma hedging in the Black and Scholes world.

5 Conclusion

We presented a simple stochastic volatility model for option pricing. We showed the consistency of our model with some well-known financial stylized facts, such as the leptokurtosis of asset returns' distributions and the smile effect of the Black and Scholes implied volatilities. Then, we analyzed option hedging problems assuming that volatilities of asset prices are stochastic.

Many empirical studies suggest a negative correlation between the stock price movements and their volatility (leverage effect). Figure 10 summarizes our results for differents values of ρ, Figure 11 details the case of a strong leverage effect.

It is important to stress the fundamental difference between the impact of stochastic volatility on the value of an option and on the value of hedge ratios *at one point in time* and the effect of stochastic volatility on the performance *over time* of different hedging schemes. For ρ strongly negative, the Black and Scholes implied volatility-based delta hedging leads systematically to an underhedged position. However, in a dynamic setting, the pure delta hedging scheme based on the Black and Scholes model outperforms the stochastic

volatility model based delta neutral strategy. This is because the one-signed Black and Scholes delta hedging bias partly corrects for the non hedge of volatility. In the case of the delta-sigma hedging strategy, the Black and Scholes hedging biases tend to swamp together and the performance of the Black and Scholes model based hedge is comparable with the performance of the delta-sigma hedging based on the "true" stochastic volatility model. In the case of delta-gamma-sigma hedging scheme, the relative performances of the two models depend on ρ. For $\rho < 0$, the stochastic volatility model based hedging substantially outperforms the strategy based on the Black and Scholes methods.

Finally, the main conclusion of this paper is the importance of volatility hedging. Hedging volatility makes hedge performances comparable to those obtained in the Black and Scholes world where the variabilities of asset prices are constant. On the other hand, failing to hedge stochastic volatility induces a risk on financial institutions' hedge positions that can not be neglected.

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A   Hedging comparison

Delta hedging									
ρ	-0.9	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.9
σ_{BS}	0.705	0.822	0.856	0.852	0.873	0.825	0.826	0.805	0.742
HP_{BS}	0.189	0.222	0.232	0.231	0.237	0.224	0.225	0.216	0.202
σ_{SV}	0.913	0.886	0.875	0.853	0.872	0.826	0.845	0.867	0.952
HP_{SV}	0.245	0.239	0.237	0.231	0.237	0.224	0.230	0.236	0.259

Delta-gamma hedging									
ρ	-0.9	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.9
σ_{BS}	0.713	0.739	0.717	0.726	0.763	0.766	0.799	0.840	0.906
HP_{BS}	0.192	0.199	0.194	0.197	0.207	0.208	0.217	0.229	0.247
σ_{SV}	1.188	1.032	0.903	0.830	0.808	0.746	0.702	0.619	0.555
HP_{SV}	0.319	0.278	0.225	0.225	0.219	0.203	0.191	0.169	0.151

Delta-sigma hedging									
ρ	-0.9	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.9
σ_{BS}	0.497	0.494	0.495	0.494	0.497	0.493	0.493	0.498	0.486
HP_{BS}	0.134	0.134	0.134	0.134	0.135	0.134	0.134	0.136	0.132
σ_{SV}	0.491	0.482	0.487	0.483	0.488	0.484	0.502	0.496	0.486
HP_{SV}	0.132	0.130	0.132	0.131	0.133	0.132	0.137	0.135	0.132

Delta-gamma-sigma hedging									
ρ	-0.9	-0.5	-0.3	-0.1	0	0.1	0.3	0.5	0.9
σ_{BS}	0.336	0.498	0.505	0.481	0.496	0.483	0.452	0.462	0.198
HP_{BS}	0.090	0.134	0.137	0.131	0.135	0.131	0.123	0.126	0.054
σ_{SV}	0.139	0.145	0.187	0.153	0.161	0.189	0.220	0.259	0.388
HP_{SV}	0.037	0.039	0.051	0.042	0.044	0.052	0.059	0.071	0.106