Efficient trading strategies with transaction costs

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January 7, 2004

Abstract

In this article, we characterize efficient contingent claims in a context of transaction costs and multidimensional utility functions. The dual formulation of utility maximization helps us outline the key notion of cyclic anticomonotonicity. Moreover, after defining a utility price in this multidimensional setting, we provide a measure of strategies inefficiency and a tool allowing to effectively compute this measure with the help of cyclic anticomonotonicity.

Keywords: cyclic anticomonotonicity, utility maximization, transaction costs, utility price.

Introduction

We consider a general multivariate financial market with transactions costs as in Kabanov ([11]), and we give tools to understand optimal strategies when agents are modelled with preferences following stochastic dominance of order 2. Precisely, an important feature of our analysis is the setting of multidimensional model of the market as well as utility functions. We provide a characterization of efficient contingent claims, i.e. chosen by agents endowed with a multidimensional utility function U. We also compute the inefficiency part of a strategy without specifying any utility function.
In the literature, these questions were studied in the case of a discrete and complete financial market by Dybvig ([4], and [5]); Jouini and Kallal

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([10]) generalizes the results in discrete markets with frictions, and when agents maximize the expected utility of their terminal wealth with respect to a numéraire. These papers show in particular the importance of the notion of anticomonotonicity, translating the intuition that efficient contingent claims are decreasing functions of Arrow-Debreu prices.
Our setting is more general as we consider a continuous financial market, with an infinite probability space, where preferences of agents are represented with the help of a multidimensional utility function, (studied in DEELSTRA and al. [6]). This multidimensional model of preferences is in accordance with the intuition that not only the liquidation value but also the holdings of the portfolio matters. Moreover, when the preferences of the agent are not only function of the liquidation value of the portfolio, the notion of anticomonotonicity is not relevant anymore. In the main results of this paper, we characterize efficient contingent claims with the notion of *cyclic anticomonotonicity*, introduced by ROCKAFELLAR ([14]); this is done with the help of the dual formulation of the problem of utility maximization. The paper is organized as follows. Section (1) presents the setting of this paper, and gives the first tools to solve our problem. Section (2) states the principal result of this paper which gives the characterization of strictly efficient contingent claims. We end this paper (section (3)) by the computation of the inefficiency size of a trading strategy.

1 The financial market

1.1 Assets and trading strategies

Let T be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space endowed with a filtration $\mathbb{F} = (F_t)_{0\leq t\leq T}$, satisfying the usual conditions. Let $S \triangleq (S^0, S^1, ..., S^d)$ be a continuous semimartingale with strictly positive components; the first component will play the role of numeraire, i.e. is assumed to be constant over time $S^0(.) = 1$.
In this market, we suppose there exists possibly constant proportional transaction costs. These transaction costs are described with a matrix $(\lambda^{ij}) \in \mathbb{M}_+^{d+1}$, where $\mathbb{M}_+^{d+1}$ is the set of square matrix with (d+1) lines with non-negatives entries. Each coefficient λ^{ij} is the proportional cost to transfer value from asset i to asset j. Furthermore, this matrix satisfies the following condition:

- $\lambda^{ii} = 0$ for all $i \in 0, ..., d$
- $(1 + \lambda^{ij}) \leq (1 + \lambda^{ik})(1 + \lambda^{kj})$ for all $i, j, k \in 0, ..., d$

These conditions translate the economic hypothesis that transaction costs can not be saved by an artificial transit. Following KABANOV([11]), we define the *solvency region* as the vectors of portfolio holdings such that the no bankruptcy condition is satisfied:

$$K \triangleq \left\{ x \in \mathbb{R}^{d+1} \mid \exists a \in \mathbb{M}_+^{d+1}, x^i + \sum_{j=0}^{d} (a^{ji} - (1+\lambda^{ij})a^{ij}) \geq 0; i = 1, ..., d \right\}$$

This closed convex cone induces a partial ordering on $\mathbb{R}^d$ as:

$$x_1 \succeq x_2 \text{ if and only if } x_1 - x_2 \in K$$

We could also introduce the positive polar associated to K, defined as

$$K^* \triangleq \left\{ y \in \mathbb{R}^{d+1} \mid \langle x, y \rangle \geq 0, \text{ for all } x \in K \right\}$$

and the partial ordering induced by K^*:

$$y_1 \succeq_* y_2 \text{ if and only if } y_1 - y_2 \in K^*$$

A trading strategy on this market is a $\mathbb{F}$-adapted, right-continuous, process L taking values in $\mathbb{M}_{d+1}$. L_t^{ij} is the cumulative net amount of funds transferred from the asset i to asset j up to the date t. Given an initial holdings vector $x \in \mathbb{R}^d$ and a strategy L, the portfolio holdings are defined by the dynamics,

$$X_t^i = x^i + \int_0^t X_s^i \frac{dS_s^i}{S_{s^-}^i} + \sum_{j=0}^{d} \left(L_t^{ji} - (1+\lambda^{ij}) L_t^{ij} \right)$$

A trading strategy is said admissible if it satisfies the no bankruptcy condition, at each time t, i.e.:

$$X^{x,L} \succeq 0$$

and we define the set of positive contingent claims attainable by an admissible strategy:

$$\mathcal{X}(x) \triangleq \left\{ X \in L^0(\mathbb{R}_+^{d+1}) \mid X = X_T^{x,L} \text{ for an admissible trading strategy L} \right\}$$

1.2 Tools of valuation and a Duality result

1.2.1 Valuation functions

In the framework of a market with transaction costs, the valuation of a portfolio with respect to a given asset is not equivalent to the valuation with respect to cash. Thus, different functions of valuation are possible. We could define the *liquidation value* of a portfolio $x_0 \in K$ as the maximum cash endowment that we can get from portfolio x_0 when clearing all the positions in risky assets and paying the transaction costs:

$$l(x) \triangleq \sup \{ w \in \mathbb{R} \mid x \geq w \mathbf{1}_1 \} .$$

This definition implies that $l(x_1) \geq l(x_2)$ if and only if $x_1 \succeq x_2$. Moreover, it is possible to reformulate the liquidation function with $K_0^* \stackrel{\Delta}{=} \{y \in K^* \mid y^0 = 1\}$:

$$l(x) = \inf_{y \in K_0^*} xy$$

To further comment on this function, we refer to KABANOV([11]), DEELSTRA and al. ([6]) and BOUCHARD ([2]).
Another function of valuation, which turns out to be very useful in our setting, is the amount of a certain position x_0 that we can get from the initial holdings vector x:

$$v_{x_0}(x) \stackrel{\Delta}{=} \sup\{w \in \mathbb{R} \mid wx_0 \preceq x\}$$

In the same way as for the liquidation function, we can give a dual formulation of $v_{x_0}(.)$ with the set $K_{x_0}^* \stackrel{\Delta}{=} \left\{y \in K^* \mid y = \frac{x_0}{\|x_0\|^2} + y^\perp\right\}$. We obtain the following proposition:

Proposition 1.1 *Let $x_0 \in int(K)$. The set $K_{x_0}^*$ is compact and K^* is the cone generated by $K_{x_0}^*$. Moreover, the amount $v_{x_0}(x)$ of the portfolio x_0 that can be obtained from the initial holdings vector x is:*

$$v_{x_0}(x) = \inf_{y \in K_{x_0}^*} yx$$

Proof of the proposition 1.1.
Let $x_0 \in int(K)$; there exists $r_0 > 0$ such that $x_\lambda = x_0 \pm x_0^\perp \in K$ as soon as $|x_0^\perp| \leq r_0$. In consequence, if we define $y_\beta = x_0 + \beta x_0^\perp$, then :

$$\begin{aligned} y_\beta x_\lambda^+ &= {}^t(x_0)x_0 + \beta^t(x_0^\perp)x_0 < 0 \text{ for } \beta > \beta_0 \\ y_\beta x_\lambda^- &= {}^t(x_0)x_0 - \beta^t(x_0^\perp)x_0 < 0 \text{ for } \beta < -\beta_0 \end{aligned}$$

We deduce that if $y_\beta \in K^*$, then $|\beta| < |\beta_0|$: the set $K_{x_0}^*$ is compact. The fact that K^* is generated by the compact set $K_{x_0}^*$ is straightforward. Now, for the last item, take $y \in K_{x_0}^*$ and $w \in \mathbb{R}$ such that $wx_0 \preceq x$. We have, by definition of K^*:

$$\langle y, x\rangle - \langle y, wx_0\rangle \geq 0$$

i.e $\langle y, x\rangle \geq w$ and we deduce that

$$v_{x_0}(x) \leq \inf_{y \in K_{x_0}^*} yx$$

To the converse inequality, if $w^* > v_{x_0}(x)$, we have $w^* x_0 \succ x$. We deduce for $y \in K_{x_0}^*$:

$$\langle y, w^* x_0\rangle = w^* \geq \langle y, x\rangle$$

This leads to :

$$v_{x_0}(x) \geq \inf_{y \in K_{x_0}^*} yx$$

$\square$.

We introduce also the function, issued from the partial ordering induced by K^*:

$$l^*(y) \triangleq \inf_{x \in K, |x|=1} xy$$

Before ending this paragraph, let us just recall the following properties:

Proposition 1.2 *The functions of valuation verify:*

- $l^*(y_1) \geq 0$ *if and only if* $y_1 \succeq 0$.
- $v_{x_0}(x_1) \geq 0$ *if and only if* $x_1 \succeq 0$.
- $\partial K^* = \{y \in K^* \mid l^*(y) = 0\}$.
- $\partial K = \{x \in K \mid l(x) = 0\}$

Proof of the proposition 1.2.
The two first items are straightforward. Item (3) and (4) can be found in DEELSTRA and al. ([6]).

1.2.2 Dual formulation of the super-replication price

In this paragraph, we give an important result of KABANOV and LAST ([12]), allowing us to write the pricing function of contingent claims with a dual formulation.
For some positive contingent claim $C \in L^0(K, \mathcal{F}_T)$, let:

$$\Gamma(C) \triangleq \left\{ x \in \mathbb{R}^{d+1} \mid X \succeq C \text{ for some } X \in \mathcal{X}(x) \right\}$$

$\Gamma(C)$ is the set of initial portfolio allowing to construct a strategy which hedges the contingent claim C. For a probability $\mathbb{Q}$ denoting $\mathcal{M}(\mathbb{Q})$ the set of all $\mathbb{Q}$-martingales, we introduce the set:

$$\mathcal{D} \triangleq \left\{ Z \in \mathcal{M}(\mathbb{Q}) \mid \frac{Z_t}{S_t} \in K^*, 0 \leq t \leq T \right\}$$

With these definitions we can state the following result.

Theorem 1.1 (KABANOV and LAST.) *Let S be a continuous process in $\mathcal{M}(\mathbb{Q})$ for some $\mathbb{Q} \sim \mathbb{P}$. Suppose further that int$(K^*) \neq \emptyset$, Then:*

$$\Gamma(C) = D(C) \triangleq \{x \in \mathbb{R}^{d+1} \mid \mathbb{E}\left(\hat{Z}_T C\right) - \hat{Z}_0 x \leq 0 \textit{ for all } Z \in D\}$$

We can therefore extend naturally the set $\mathcal{D}$. Let

$$\mathcal{Y}(y) = \left\{Y \in L^0(K^*, \mathcal{F}_T) \mid \mathbb{E}\,(YX) \leq xy \text{ for all } x \in K \text{ and } X \in \mathcal{X}(x)\right\}$$

With this result of duality, we can evaluate the amount of portfolio x_0 needed to hedge the contingent claim X_0. Indeed, define

$$\mathcal{D}^{\perp} = \left\{Y \in \mathcal{Y}(y) \text{ for } y \in K^*_{x_0}\right\}$$

We have:

Lemma 1 *The amount of portfolio x_0 in order to hedge the contingent claim x_0 is given by:*

$$\pi(X_0, x_0) \triangleq \sup_{Y \in \mathcal{D}^{\perp},\, y \in K^*_0} \mathbb{E}\,(YX)$$

Proof of the lemma 1.
Indeed, since K^*_0 generates K^*, we have that X_0 can be hedged by the initial holdings vector λx_0 if and only if:

$$\mathbb{E}\,(YX) \leq \lambda x_0 y_0 = \lambda \text{ if and only if } Y \in \mathcal{D}^{\perp}(x_0)$$

i.e.

$$\sup_{Y \in \mathcal{D}^{\perp},\, y \in K^*_0} \mathbb{E}\,(YX) \leq \lambda$$

□.

A straightforward implication is that $X \in \mathcal{X}(x_0)$ if and only if $\sup_{Y \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\,(XY) \leq 1$.
We end this section by an interesting result, proved in Deelstra and al. ([6]), which give a characterization of attainable contingent claims.

Lemma 2 *Let S be a continuous process in $\mathcal{M}(\mathbb{Q})$ for some $\mathbb{Q} \sim \mathbb{P}$ and suppose that int$(K^*) \neq \emptyset$. Let $X_0 \in L^0(K, \mathcal{F}_T)$ and $x_0 \in K$ be such that:*

$$\sup_{y \in K^*} \sup_{Y \in \mathcal{Y}(y)} \mathbb{E}(X_0 Y) - x_0 y = \mathbb{E}(X_0 Y_0) - x_0 y_0 = 0$$

for some $y_0 \in K^*$ and $Y_0 \in \mathcal{Y}(y_0)$ with $\mathbb{P}\,[Y_0 = 0] = 0$. Then the contingent claim is attainable from the initial wealth x_0.

2 Characterization of strictly efficient trading strategies

In a complete and discrete market, where prices of every contingent claims depend on a vector of density price of states, Dybvig([4] show that efficient contingent claims are decreasing functions of the vector of density prices. In particular, studying the problem in a market with frictions in a finite space,

JOUINI and KALLAL ([10]) prove that this property of *anticomonotonicity* can be seen as a derivation of the first order condition in the convex optimization problem of maximizing the expected utility of terminal wealth.
In a market with transactions costs, the value of a portfolio is not equivalent to its liquidation value, and it is restrictive to assume that the agents maximize the expected utility of the liquidation of terminal portfolio. It is more relevant to consider each agent endowed with a **utility function** $U \in \mathcal{U}$, being the set of functions mapping $\mathbb{R}^{d+1}$ into $\mathbb{R}$ with effective domain $int(K) \subset dom(U) \subset K$, and satisfying the conditions:

- U is strictly increasing on K, i.e $U(x_1) > U(x_2)$ for all $x_1 \succ x_2$.
- U is concave on K

Each agent chooses an optimal strategy, depending on his preferences (i.e. on the utility function U) and on his initial portfolio x_0. This optimal strategy is chosen in order to maximize the expected utility of the terminal holdings vector.

Definition 2.1 *A strategy L, with a positive initial portfolio $x_0 \in int(K)$, is strictly efficient if and only if there exists an utility function U such that $X_T^{x_0,L} \in \mathcal{X}(x_0)$ is solution to the problem of maximization:*

$$\mathbb{E}\left(U(X_T^{x,L})\right) = \sup_{X \in \mathcal{X}^1(x)} \mathbb{E}\left(U(X)\right)$$

where

$$\mathcal{X}^1(x) \triangleq \mathcal{X}(x) \cap L^1(K, \mathcal{F}_T)$$

In order to avoid problems of definition of expected utility for some contingents claims and utility functions $U \in \mathcal{U}$, we work with integrable contingent claims : indeed, our approach is to use minimal technical hypothesis for the agents, and the use of the space $L^0(K, \mathcal{F}_T)$ implies that the expected utility for some contingent claims may be not well defined.
As it was already the case in JOUINI and KALLAL ([10], the basic idea is to characterize strictly efficient strategies with results of duality. But in a context of continuous probability space, duality is more complex to handle. Maximizing the expected utility in an incomplete market was studied in a very large setting by KRAMKOV and SCHACHERMAYER ([13]. HUGONNIER and KRAMKOV ([9]) study this problem with a random endowment at the final date. In a multidimensional case, the problem was studied by DEELSTRA and al.([6]). Nevertheless, these articles focus on some suitable hypothesis on the utility function, as the one on the asymptotic elasticity, in order to have a solution for the primal problem. In our setting, we are not interested in this question of existence and thus we don't need to impose any regularity hypothesis on the utility function.

In the following paragraph, we present more precisely the notion of anticomonotonicity and introduce *cyclic anticomonotonicity*, which is more relevant in our setting. Then, we present our principal result, which characterizes strictly efficient contingent claims.

2.1 Anticomonotonicity

In the one-dimensional theory of decision, and in the case of stochastic dominance of order 2, anticomonotonicity between a random variable of pricing Y and a contingent claim X derives from duality and the fact that $Y \in \lambda\partial U(X)$, for $\lambda > 0$ (see [10]). Let us recall this notion:

Definition 2.2 *Two random variables X_1 and X_2 defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are anticomonotonic if there exists A in $\mathcal{F}$, with probability one, such that:*

$$\left[X_1(\omega) - X_1(\omega')\right]\left[X_2(\omega) - X_2(\omega')\right] \leq 0 \text{ for all } (\omega, \omega') \in A \times A.$$

Thus, one can think first that we need the natural extension of one-dimensional anticomonotonicity to the multidimensional framework: i.e., two random vectors X and Y of dimension $d+1$ are said anticomonotonic if and only there exists a measurable set A of probability 1 such that
$\left(\forall(\omega_1, \omega_2) \in A^2\right)(\langle X(\omega_1) - X(\omega_2), Y(\omega_1) - Y(\omega_2)\rangle \leq 0)$.
But it turns out in the multidimensional case that this is not enough to characterize strictly efficient contingent claims. We use the following notion:

Definition 2.3 *Two random vectors X and Y of dimension $d+1$ are said cyclically anticomonotonic if and only there exists a measurable set A of probability 1 such that:*

$$(\forall p \geq 2, \forall(\omega_1, \omega_2, ..., \omega_p) \in A^p)$$
$$(\langle X(\omega_1), Y(\omega_1) - Y(\omega_2)\rangle + \langle X(\omega_2), Y(\omega_2) - Y(\omega_3)\rangle + ... + \langle X(\omega_p), Y(\omega_p) - Y(\omega_1) \leq 0)$$

Indeed, more or less, in our setting, we deduce that for a strictly efficient contingent claim $X_0 \in \mathcal{X}(x_0)$, there exists, in the same way as the uni-dimensional framework, a random vector of pricing $Y_0 \in \mathcal{Y}$ such that $Y_0 \in \lambda\partial U(X_0)$. Rockafellar ([14]) show that this is in fact characterized by the *cyclical anticomonotonicity* .
Before continuing this discussion, we would like to stress a little subtlety on the notion of "almost surely". Indeed, we could have defined anticomonotonicity with existence of measurable sets A of probability 1 on the **product** space. This is in fact equivalent:

Proposition 2.1 *Let X and Y two random vectors on the space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $p \in \mathbb{N}^*$, define $\mathbb{P}^{\otimes p}$ as the usual propability product on the produce* *space $\Omega^{\otimes p}$. The cyclic anticomonotonicity between X and Y is equivalent to:*

$$(\forall p \geq 2) \left(\exists A \in \Omega^{\otimes^p} \mid \mathbb{P}^{\otimes^p}(A) = 1\right) (\forall (\omega_1, ..., \omega_p) \in A)$$
$$(\langle X(\omega_1), Y(\omega_1) - Y(\omega_2)\rangle + \langle X(\omega_2), Y(\omega_2) - Y(\omega_3)\rangle + ... + \langle X(\omega_p), Y(\omega_p) - Y(\omega_1) \leq 0)$$

Proof of the proposition 2.1.
It is straightforward that cyclic anticomonotonicity implies the property of the proposition. Conversely, assume that X and Y verify this property. It is enough to prove there exists X^* and Y^* with $X^* \overset{a.s.}{=} X$ and $Y^* \overset{a.s.}{=} Y$, such that X^* and Y^* are cyclically anticomonotonic. Denoting $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$, we define the set for each vector of $I \in \mathbb{N}^{d+1}_{n2^n}$:

$$A_I = \bigcap_{0 \leq l \leq d} \left\{ \frac{I(l)}{2^n} \leq X_l < \frac{I(l)+1}{2^n} \right\}$$
$$B_K = \bigcap_{0 \leq l \leq d} \left\{ \frac{I(l)}{2^n} \leq Y_k < \frac{I(l)+1}{2^n} \right\}$$

Step 1 - construction of version X.
We define

$$X^n_l = \sum_{(I,K) \in P_n} \frac{I(l)}{2^n} \mathbf{1}_{A_I \bigcap B_K}$$

with

$$P_n = \left\{(I, K) \in \left(\mathbb{N}^{d+1}_{n2^n}\right)^2 \mid \mathbb{P}\left(A_I \cap B_K\right) \neq 0\right\}$$
$$N_n =^C \left(\bigcup_{(I,K) \in P_n} A_I \bigcap B_K \right)$$

If $\omega \notin \bigcup N_n$, we have $|X^n - X| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$; for all $\omega \in \Omega$, X^n converge and we define:

$$X^* \triangleq \lim_{n \to +\infty} X^n$$

and $X^* \overset{p.s.}{=} X$, since $\mathbb{P}(N_n) \to 0$.
Step 2 - Construction of version Y^*.
In the same way, we define:

$$Y^n_l = \sum_{(I,K) \in P_n} \frac{K(l)}{2^n} \mathbf{1}_{A_I \bigcap B_K}$$

If $\omega \notin \bigcup N_n$, we have $|Y^n(\omega) - Y(\omega)| \leq \frac{1}{2^{n+1}}$ for all $n \in \mathbb{N}$. Therefore, for all ω, Y^n converge and we define:

$$Y^* \triangleq \lim_{n \to +\infty} Y_n$$

and $Y^* \overset{p.s.}{=} Y$
The sequence Y^n and X^n have been constructed in order to be almost cyclically anticomonotonic; indeed, let $\Omega_n = \Omega \setminus N_n$ and $p \in \mathbb{N}^*$. We have:

$$(\forall (\omega_1, ..., \omega_p) \in (\Omega_n)^p) \; \langle Y^n(\omega_1), X^n(\omega_1) - X^n(\omega_2)\rangle + ... + \langle Y^n(\omega_p), X^n(\omega_p) - X^n(\omega_1)\rangle \quad -$$
$$\langle Y(\omega_1), X(\omega_1) - X(\omega_2)\rangle + ... + \langle Y(\omega_p), X(\omega_p) - X(\omega_1)\rangle \quad \leq \quad 3p\frac{n}{2^n}$$

Therefore:

$$(\forall (\omega_1,...,\omega_p)\in(\Omega_n)^p)\left(\langle Y^n(\omega_1),X^n(\omega_1)-X^n(\omega_2)\rangle+...+\langle Y^n(\omega_p),X^n(\omega_p)-X^n(\omega_1)\rangle<3p\frac{n}{2^n}\right)$$

We conclude that if we define $A=\Omega\setminus\cap_{n\geq 0}N_n$, we have:

$$(\forall (\omega_1,...,\omega_p)\in(A)^p)\left(\langle Y^*(\omega_1),X^*(\omega_1)-X^*(\omega_2)\rangle+...+\langle Y^*(\omega_p),X^*(\omega_p)-X^*(\omega_1)\rangle\leq 0\right)$$

An important corollary of this proposition is:

Corollary 2.1 *Let $d\in\mathbb{N}^*$ and $(\Omega,\mathcal{F},\mathbb{P})$ a probability space. Suppose X and Y are two vector of dimension $d+1$ such that we can't find a $\varepsilon>0$ and some non negligeable measurable sets $\Omega_1,\Omega_2,...,\Omega_n$ verifying the property:*

$$(\forall (\omega_1,...,\omega_n)\in\Omega_1\times...\times\Omega_n)\left(\langle Y(\omega_1),X(\omega_1)-X(\omega_2)\rangle+...+\langle Y(\omega_n),X(\omega_n)-X(\omega_1)\rangle\geq\varepsilon\right)$$

Then X and Y are cyclically anticomonotonic.

Indeed, it gives a criterium very tractable to prove that two random vectors X and Y are cyclically anticomonotonic.

Finally, in the one dimensional case, on discrete probability spaces with equiprobability states or on probability spaces atomless, we can find, for any two fixed distributions F_X and F_Y, random variables X distributed as F_X, and Y distributed as F_Y such that X and Y are anticomonotonic (see [7]). The next result proves that this result is extendable to the multidimensional case for the notion of cyclic anticomonotoncity.

Proposition 2.2 *Let $X\in\mathcal{X}^1$ and $Y\in\mathcal{Y}$. There exist a random vector $\tilde{X}$ and a random vector $\tilde{Y}$, with respectively the same distribution as X and Y, such that $\tilde{X}$ and $\tilde{Y}$ are cyclically anticomonotonic.*

Proof of the proposition 2.2.
In the following, we denote C_X a copula of X, and C_Y a copula of Y. Consider the set $\mathcal{C}(X,Y)$ of copula in $\mathbb{R}^{2d+2}$, such that for all $C\in\mathcal{C}(X,Y)$, the marginal copula of the $d+1$ first variables is the copula C_X, and the marginal copula of the $d+1$ last variables is the copula of C_Y:

$$\begin{aligned} C(u_1,u_2,...,u_{d+1},1,...,1) &= C_X(u_1,u_2,...,u_{d+1}) \\ C(1,...,1,v_1,v_2,...,v_{d+1}) &= C_Y(u_1,u_2,...,u_{d+1}) \end{aligned}$$

It is straightforward that the set $\mathcal{C}(X,Y)$ is closed with respect to the topology of convergence simple in $\mathcal{C}$, the set of copula defined on $[0,1]^{2d+2}$. But the set of copula $\mathcal{C}$ is compact with respect to this topology (see [3], theorem 2.3, and [1] for the convergence of probability measures). Let X_n a sequence of random vectors distributed as X such that $\lim_{n\to+\infty}\mathbb{E}\left(YX_n\right)=\inf\{\mathbb{E}(YX)\mid X \text{ is distributed as } X_0\}$. With maybe an extraction, we could suppose that the sequence of copula $C_n\in\mathcal{C}(X,Y)$

of the random vector (X_n, Y) converges to a copula $C \in \mathcal{C}(X,Y)$. Define then the random vector $\tilde{X}$ with the same distribution as X_0, such that C is a copula of $(\tilde{X}, Y)$. By construction, $\langle X_n, Y\rangle$ converges in law to $\langle \tilde{X}, Y\rangle$, and therefore:

$$\mathbb{E}(\langle \tilde{X}, Y\rangle) \leq \lim_{n\to+\infty} \mathbb{E}\left(\langle X_n, Y\rangle\right)$$

Now, let's prove that $\tilde{X}$ and Y are cyclically anticomonotonic. If it is not the case, we deduce from the corollary (2.1) the existence of $\varepsilon > 0$, $n > 0$ and some non negligeable sets $\Omega_1, ..., \Omega_n$ such that:

$$(\forall(\omega_1, .., \omega_n) \in (\Omega_1 \times ... \times \Omega_n)) \left(\langle \tilde{X}(\omega_1) - \tilde{X}(\omega_2), Y(\omega_1)\rangle + ... + \langle \tilde{X}(\omega_n) - \tilde{X}(\omega_1), Y(\omega_n)\rangle \geq \varepsilon\right)$$

and, since we have assumed that our space is atomless, we can always choose the sets $\Omega_1, ..., \Omega_n$ with the same probability p. Define the random vector X^*, distributed as X_0 with:

$$\begin{cases} X^*_{|\Omega\backslash\Omega_1\cup...\cup\Omega_n} = \tilde{X}_{|\Omega\backslash\Omega_1\cup...\cup\Omega_n} \\ X^*_{|\Omega_i} \text{ distributed as } \tilde{X}_{|\Omega_{i+1}} \text{ for } i < n \\ X^*_{|\Omega_n} \text{ distributed as } \tilde{X}_{|\Omega_1} \end{cases}$$

In consequence, we have

$$\mathbb{E}(\langle X^*, Y\rangle) - \mathbb{E}(\langle \tilde{X}, Y\rangle) = \sum_{i=1}^{n} \mathbb{E}\left[\langle Y, (X^* - \tilde{X})_{|\Omega_i}\rangle\right]$$

But, by construction, we have $\sum_{i=1}^{n} \langle Y, (X^* - \tilde{X})_{|\Omega_i}\rangle \leq \varepsilon$, which implies

$$\mathbb{E}(\langle X^*, Y\rangle) - \mathbb{E}(\langle \tilde{X}, Y\rangle) \leq -p\varepsilon$$

This is a contradiction and conclude the proof □.

2.2 Theorem of characterization of strictly efficient strategies

We turn now to the problem of characterizing solutions of:

$$\sup_{X\in\mathcal{X}^1(x_0)} \mathbb{E}\left(U(X)\right)$$

for some utility function U and $x_0 \in int(K)$. As we have already pointed out, in markets with transactions costs, it is not only the liquidation value of the portfolio x_0 which matters, but also the whole holdings vector. In consequence, our problem has a natural direction, and we dualize our problem with respect to this direction x_0, i.e. we consider the perturbed problem $u(\lambda x_0)$ on the line $\mathbb{R}_*^+ x_0$:

$$\begin{aligned} u_{x_0}(\lambda) &= \sup_{X\in\mathcal{X}^1(\lambda x_0)} \mathbb{E}\left(U(X)\right) \\ &= \sup_{X\in\mathcal{X}^1} \inf_{\beta>0, Y\in\mathcal{D}^\perp(x_0)} \mathbb{E}\left(U(X)\right) - \beta(\mathbb{E}[XY] - \lambda) \end{aligned}$$

We then associate the following dual problem:

$$
\begin{aligned}
v_{x_0}(\beta) &= \inf_{Y \in \mathcal{D}^{\perp}(x_0)} \sup_{X \in \mathcal{X}^1} \mathbb{E}\left[U(X)\right] - \beta \mathbb{E}[XY] \\
&= \inf_{Y \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\left[V(\beta Y)\right]
\end{aligned}
$$

where V is defined as the Legendre-Fenchel transform of U:

$$
V(Y) = \sup_{X \in K} U(X) - XY
$$

We want to prove now that our dual problem has a solution and that dual and primal problem have the same value. For this, we need:

Lemma 3 . *Let $x_0 \in int(K)$ and $\beta > 0$. The family $(V(\beta Y))^{-}$ for $Y \in \mathcal{D}^{\perp}(x_0)$ is uniformly integrable.*

Proof of the lemma 3.
Let $\varepsilon > 0$ and choose a vector x^* such that $0 \prec x^* \prec \frac{\varepsilon}{2} x_0$. $x^* \in int(K)$, and therefore belongs to the effective domain of U. If we choose $y^* \in \partial U(x^*)$, by usual theory of conjugate functions, we have, $x^* \in \partial V(y^*)$, which implies:

$$
V(y)^{-} \leq \left(V(y^*) + \langle x^*, y - y^* \rangle\right)^{-} \leq V(y^*)^{-} + \langle x^*, y \rangle
$$

We deduce first that:

$$
\sup_{Y \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\left(V(Y)\right) \leq V(y^*)^{-} + \sup_{Y \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\left(x^* Y\right) < +\infty
$$

With Tchebychev, we choose $a \in \mathbb{R}_+^*$ such that $V(y^*)\mathbb{P}\left(V(Y) \geq a\right) = \frac{\varepsilon}{2}$. We conclude:

$$
\begin{aligned}
\sup_{Y \in \mathcal{D}^{\perp}(x_0)} \int_{V(Y) \geq a} V(Y)^{-} d\mathbb{P} &\leq \sup_{Y \in \mathcal{D}^{\perp}(x_0)} V(y^*)^{-} \mathbb{P}\left(V(Y) \geq a\right) + \mathbb{E}\left(x^* Y\right) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

$\square$.

This uniformly integrability property is a key result to prove the following lemma which tells us there is no gap between the value of the primal and dual problem.

Lemma 4 *Let U and x_0 such that $u_{x_0}(1) < +\infty$. Then:*

$$
v_{x_0}(\beta) = \sup_{\lambda > 0} \left[u_{x_0}(\lambda) - \lambda \beta\right]
$$

Moreover there exists $Y_0 \in \mathcal{D}^{\perp}(x_0)$ solution to the dual problem as soon as $v_{x_0}(\beta) < +\infty$

Proof of the lemma 4.
The following proof is slightly adapted from the proof of KRAMKOV and SCHACHERMAYER ([13]). Consider the set $\mathcal{D}^{\perp}(x_0)$ and define, for $n > 0$, the sets $\mathcal{B}_n$ to be the positive elements of the ball of radius n of L^{∞}, i.e., $\mathcal{B}_n \triangleq \{g \mid 0 \leq |g| \leq n\}$. The sets $\mathcal{B}_n$ are $\sigma(L^{\infty}, L^1)$-compact, and since $\mathcal{D}^{\perp}(x_0)$ is closed-convex subset of L^1, we obtain:

$$\sup_{g \in \mathcal{B}_n} \inf_{h \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\left[U(g) - \beta g h\right] = \inf_{h \in \mathcal{D}^{\perp}(x_0)} \sup_{g \in \mathcal{B}_n} \mathbb{E}\left[U(g) - \beta g h\right]$$

Moreover, since $g \in \mathcal{X}(x_0)$ if and only if:

$$\sup_{h \in \mathcal{D}^{\perp}(y_0)} \mathbb{E}\left[g h\right] \leq 1$$

we have:

$$\lim_{n \to +\infty} \sup_{g \in \mathcal{B}_n} \inf_{h \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\left[U(g) - \beta g h\right] = \sup_{\lambda > 0} u(\lambda) - \lambda \beta$$

On the other hand,

$$\inf_{h \in \mathcal{D}^{\perp}(x_0)} \sup_{g \in \mathcal{B}_n} \mathbb{E}\left[U(g) - \beta g h\right] = \inf_{h \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}[V_n(\beta h)) \triangleq v^n(\beta)$$

Consequently, it is sufficient to prove that $\lim_{n \to +\infty} v^n(\beta) = v(\beta)$. Evidently, we have $v_n \leq v$. Let $(h_n)_{n \geq 1}$ be a sequence in $\mathcal{D}^{\perp}(x_0)$ such that:

$$\lim_{n \to +\infty} \mathbb{E}\left[V_n(\beta h^n)\right] = \lim_{n \to +\infty} v^n(\beta)$$

Since the set $\mathcal{D}^{\perp}(x_0)$ is bounded in L^1, we can apply the lemma of KOMLÒS (see [8]) and find a sequence $f^n \in conv(h^n, h^{n+1}, ...)$, which converges almost surely to a random vector h. We have $h \in \mathcal{D}^{\perp}(x_0)$ by closure of $\mathcal{D}^{\perp}(x_0)$ under convergence in probability.
However, from the convexity of the function V^n, we have:

$$\mathbb{E}\left[V^n(\beta f^n)\right] \leq \sup_{m \geq n} \mathbb{E}\left[V^m(\beta h^m)\right]$$

Moreover, **the family $V^n(f^n)^-$ is uniformly integrable**. Indeed, let $y_0 \in K^*$ such that $\partial V(y_O) = \partial V_n(y_0) = \partial V_1(y_0)$. Since the mapping $\partial V(.)$ is cyclically comonotonic, $\partial V_n(y) = \partial V(y)$ as soon as $l(y) \geq l(y_0)$, i.e. $V_n(y) = V(y)$ as soon as $l(y) \geq l(y_0)$. We deduce then the property of uniform integrability with the one of of $(V(\beta f^n))^-$.
This leads to the following inequalities, with the Fatou's Lemma:

$$\lim_{n \to +\infty} \mathbb{E}\left[V_n(\beta h_n)\right] \geq \liminf_{n \to +\infty} \mathbb{E}\left[V^n(f^n)\right] \geq \mathbb{E}\left[V(h)\right] \geq v(y)$$

which proves the equality. □.

With this result, we can give a simple characterization of strictly efficient strategies, which is one of our principal results:

Theorem 2.2 (Characterization of strictly efficient contingent claims). *A contingent claim X_0 is strictly efficient if and only if there exists an $Y_0 \in \mathcal{Y}(y_0)$ for some $y_0 \in K^*$, such that:*

- $\mathbb{P}(Y_0 = 0) = 0$
- $\mathbb{E}(X_0 Y_0) = x_0 y_0$.
- *Random variables X_0 and Y_0 are cyclically anticomonotonic.*
- $$\left\{ \begin{array}{rcl} (\exists i \in \{0, ..., d\}) \left(\sup ess_{\omega\in\Omega} Y_0^i(\omega) = +\infty\right) & \Rightarrow & \inf ess_{\omega\in\Omega} l\left(X_0(\omega)\right) = 0 \\ (\exists i \in \{0, ..., d\}) \left(\inf ess_{\omega\in\Omega} Y_0^i(\omega) > 0\right) & \Leftarrow & \sup ess_{\omega\in\Omega} l\left(X_0(\omega)\right) < +\infty \end{array} \right.$$

Proof of the theorem 2.2.
First implication: Let X_0 a contingent claim strictly efficient with the utility function U, and x_0 the initial portfolio. The function $\lambda \mapsto u(\lambda)$ is concave, and let $\beta_0 \in \partial u(1)$. From the lemma 4, v is the Legendre-Fenchel transform of U, and we have:

$$v(\beta_0) = u(1) - \beta_0$$

This implies in particular $v(\beta_0) < +\infty$ and the existence of $Y_0 \in \mathcal{D}^{\perp}(x_0)$ solution to the dual problem. We deduce that

$$\begin{array}{rcl} u(1) = \mathbb{E}(U(X_0)) & \leq & \mathbb{E}\left(V(\beta_0 Y_0) + \beta_0 Y_0 X_0\right) \\ & \leq & v(\beta_0) + \beta_0 \end{array}$$

The above inequalities become equalities and lead to:

$$X_0 = argmax_{X\in\mathcal{X}^1}\mathbb{E}\left(U(X) - XY_0\right) \quad (1)$$
$$\mathbb{E}(X_0 Y_0) = x_0 y_0 \quad (2)$$

$\mathbb{P}(Y_0 = 0) = 0$. Suppose the existence of a set A, with probability not equal to zero, such that $(Y_0)_{|A} = 0$. Define $\tilde{X} = X_0 + \mathbf{1}_A$, with. We have, by strict increasing of U,

$$\mathbb{E}\left(U(\tilde{X})\right) > \mathbb{E}\left(U(X_0)\right)$$

On the other hand, $\mathbb{E}\left(Y_0\tilde{X}\right) = \mathbb{E}\left(Y_0 X_0\right)$, which is in contradiction with (1).
Y_0 and X_0 are cyclically anticomonotonic. Indeed, with (1), there exists a probabtility set A of measure 1 such that:

$$Y_0(\omega) \in \partial U\left(X_0(\omega)\right), \forall \omega \in A$$

which is a characterization of cyclic anticomonotonicity (see ROCKAFELLAR,[14]), and give us the result.
$(\exists i \in \{0, ..., d\}) \left(\sup ess_{\omega\in\Omega} Y_0^i(\omega) = +\infty\right) \Rightarrow \inf ess_{\omega\in\Omega} l\left(X_0(\omega)\right) = 0$.
Suppose on the contrary there exists $i \in \{0, ..., d\}$ with $\sup ess_{\omega\in\Omega} Y_0^i(\omega) = +\infty$ and $\inf ess_{\omega\in\Omega} l\left(X_0(\omega)\right) > 0$. Choose $\varepsilon > 0$ such that for all $\omega \in \Omega$, $X_0(\omega) \succ 2\varepsilon 1_i$. Since $Y_0 \in \partial U(X_0)$ a.s., we have:

$$U(\varepsilon) - U(X_0) \leq \langle Y_0, \varepsilon - X_0\rangle \leq \langle Y_0, \varepsilon 1_i - 2\varepsilon 1_i\rangle = -\langle Y_0, \varepsilon 1_i\rangle$$

and the left side of the inequality tends to $-\infty$, i.e. $\varepsilon \notin dom(U)$, which is in contradiction with our hypothesis.

$(\exists i \in \{0,...,d\}) \left(\inf ess_{\omega\in\Omega} Y_0^i(\omega) > 0\right) \Leftarrow \sup ess_{\omega\in\Omega} l\,(X_0(\omega)) < +\infty$.
Indeed, if it isn't the case, we could choose $\hat{x} \succ x^*$ with $x^* \succeq X_0(\omega)$ for all $\omega \in \Omega$. We deduce, since $Y_0 \in \partial U(X_0)$:

$$U(\hat{x}) - U(x^*) \leq U(\hat{x}) - U(X_0) \leq \langle Y_0, \hat{x} - X_0 \rangle \leq \langle Y_0, \hat{x} \rangle$$

and the last term of this inequality tends to 0. We conclude that $U(\hat{x}) = U(x^*)$, which is in contradiction with the hypothesis of strict increasing of the utility function U.
Second implication : Let X_0 and Y_0 which verify the properties of the theorem. Let A the measurable set of probability one given by the definition of cyclical anticomonotonicity. We fix an $\omega_0 \in A$ and $(X_0^*, Y_0^*) \triangleq X_0(\omega_0), Y_0(\omega_0)$. We define the utility function U on K by:

$$U(x) = \inf \left\{ \langle x - X_0(\omega_m), Y_0(\omega_m) \rangle + ... + \langle X_0(\omega_1) - X_0(\omega_0), Y_0(\omega_0) \rangle \right\}$$

where the infinimum is taken over all finite sets $(\omega_0, \omega_1, ..., \omega_m)$ (m arbitrary) of elements of A.
U is a proper closed concave function. Since U is an infinimum of a certain collection of affine functions, U is a closed concave function. Moreover, $U(X_0(\omega)) = 0$ by cyclic anticomonotonicity of X_0 and Y_0 and hence U is proper.
For each $\omega \in A$, $Y_0(\omega) \in \partial U\,(X_0(\omega))$. Indeed, Let $\omega \in A$, it is enough to show that for any $\alpha > U\,(X_0(\omega))$, and any $z \in \mathbb{R}^{d+1}$, we have:

$$U(z) < \alpha + \langle Y_0(\omega), z - X_0(\omega) \rangle$$

Indeed, by definition of U, there exists some ω_i, $i = 1, ..., m$ such that:

$$\alpha > \langle Y_0(\omega_m), X_0(\omega) - X_0(\omega_m) \rangle + ... + \langle Y_0(\omega_0), X_0(\omega_1) - X_0(\omega_0) \rangle$$

We deduce, by definition of U and setting $\omega_{m+1} = \omega$:

$$U(z) \leq \langle Y_0(\omega_m), z - X_0(\omega_{m+1}) \rangle + ... + \langle Y_0(\omega_0), X_0(\omega_1) - X_0(\omega_0) \rangle < \alpha + \langle Y_0(\omega_m), z - X_0(\omega) \rangle$$

and this proves that $Y_0(\omega) \in \partial U(X_0(\omega))$.
U is increasing with respect to $\succeq$. We prove first $U(\hat{z}) \geq U(z)$ as soon as $\hat{z} \succeq z$. Indeed, let $\varepsilon > 0$ and $(z_0, y_0), ..., (z_m, y_m)$ such that:

$$\langle \hat{z} - z_m, y_m \rangle + ... + \langle z_1 - z_0, y_0 \rangle \leq U(\hat{z}) + \varepsilon$$

Therefore:

$$U(z) \leq \langle z - z_m, y_m \rangle + ... + \langle z_1 - z_0, y_0 \rangle \leq U(\hat{z}) + \varepsilon + \langle z - \hat{z}, y_m \rangle$$

and since $\hat{z} - z \in K$ and $y_m \in K^*$, we have:

$$U(z) \leq U(\hat{z}) + \varepsilon$$

and this, for all $\varepsilon > 0$.
$int(K) \subset dom(U)$: first, since we have always $dom(U) \subset dom\,(\partial U(.))$, U is finite on each $X_0(\omega)$ for $\omega \in \Omega$. Suppose first that $\inf_{\omega\in\Omega} essl(X) = 0$. Let z be in $int(K)$. We can choose an $\omega_0 \in \Omega$, with $X_0(\omega_0) \preceq z$, and deduce that $U\,(X_0(\omega_0)) \leq U(z)$, i.e. $z \in dom(U)$. If, on the contrary, $\inf_{\omega\in\Omega} essl(X) > 0$, then, by hypothesis

$\sup_{\omega\in\Omega} essl^*(Y) < +\infty$. This leads to the result.
U is strictly increasing. We take $\hat{z}$ and z such that $\hat{z} \succ z$.
First, suppose that, a.s. $X_0(\omega) \preceq \hat{z}$. In particular $\sup ess_{\omega\in\Omega} l\left(X_0(\omega)\right) < +\infty$, and from the third item, define $\varepsilon = \frac{1}{2|\hat{z}-z|} \inf_{\omega\in\Omega} l^*\left(Y_0(\omega)\right) > 0$. In the same way, we can choose y_m such that :

$$U(z) \leq U(\hat{z}) + \varepsilon + \langle z - \hat{z}, y_m \rangle$$

Therefore, by construction $\langle \hat{z} - z, y_m \rangle > \varepsilon$ and

$$U(z) < U(\hat{z})$$

Now, suppose there exists ω_0 such that $X_0(\omega_0) \succ \hat{z}$. We choose $\hat{z} \succ z_1 \succ z$ with $z_1 \in \mathbb{R}\left(X_0(\omega_0) - \hat{z}\right)$. We have:

$$U(z_1) \geq U(z) + \langle Y_0(\omega_0), z_1 - z \rangle$$

But $Y_0(\omega_0) \in int(K^*)$ and in consequence $\langle Y_0(\omega_0), z_1 - z \rangle > 0$; we conclude that $U\left(\hat{z}\right) \geq U\left(z_1\right) > U\left(z\right)$.
X_0 is strictly efficient for U. Let $X \in \mathcal{X}(x_0)$. Since, $Y_0 \in \partial U(X_0)$ almost surely, we have:

$$\mathbb{E}\left(U(X)\right) - \mathbb{E}((U(X_0)) \leq \mathbb{E}(Y_0(X - X_0)) \leq 0$$

Therefore X_0 is efficient for U $\square$.

This theorem give us immediately that the set of strictly contingent claims is a cone, as it was the case in the discrete setting (see [10]). However, as Kramkov and Schachermayer (see [13]) remark, there could be here a "loss of mass", i.e. $\mathbb{E}\left((Y_0)_1\right) \neq ((y_0)_1)$ and the sum of a strictly contingent claim with some cash is not necessarily a strictly contingent claim. This result, which is anti-intuitive, tell us that investing in the cash can change the strict efficiency of a strategy. Another difference with the discrete setting is the third condition of the theorem; $\sup ess_{\omega\in\Omega} l(X_0) < +\infty$ implying a condition on the pricing vector Y comes from the fact the slope of utility function must be strictly positive on all his domain, even out of the support of X_0. One can see also that the other condition is linked to a question of support of the utility function. However, we stress that $int(K) \subset dom(U)$ is a key hypothesis for the lemma (3), which is essential to prove theorem (4).

3 Inefficiency size

In this section, we study the quality of a strategy, leading to $X_0 \in \mathcal{X}^1$, whatever the preferences of the agent are and quantify the eventual inefficiency. To introduce our first notion, let us note that a strictly efficient contingent claim X_0 for an utility function $U \in \mathcal{U}$ with respect to the initial portfolio x_0 can be characterized as the solution of the problem of optimization:

$$\inf_{X\in\mathcal{B}^U(X_0)} \pi(X, x_0)$$

where we have defined the set $B^U(X_0)$ as the set of contingent claims preferred to X_0 by an agent with a utility function U:

$$B^U(X_0) = \left\{X \in \mathcal{X}^1 \mid \mathbb{E}\left(U(X)\right) \geq \mathbb{E}\left(U(X_0)\right)\right\}$$

Since we now want studying the efficiency whatever the preferences of the agent, this leads to the following notion:

Definition 3.1 *Let an admissible trading strategy L with initial value x_0 which leads to the contingent claims X_0. The utility price of the contingent claim X_0 with respect to the initial portfolio x_0 is defined as the percentage of the portfolio x_0 for any agent to find a strategy giving the same expected utility as X_0 :*

$$P^{\mathcal{U}}(X_0, x_0) \triangleq \sup_{U \in \mathcal{U}} \inf_{X \in \mathcal{B}^U(X_0)} \pi(X, x_0)$$

The inefficiency size is then $I^{\mathcal{U}}(X_0, x_0) = 1 - P^{\mathcal{U}}(X_0, x_0)$

Our goal is to use results of duality to evaluate the inefficiency size with the difference of the liquidated price of X_0 and a certain contingent claim $\tilde{X}$ well chosen. It is not straightforward that this contingent claim exists and the question of existence is one part of our matter.
A natural set in this setting to consider is the set $\mathcal{P}(X_0)$ of contingent claims preferred to X_0 by any agent:

$$\mathcal{P}(X_0) = \left\{X \in \mathcal{X}^1 \mid \mathbb{E}\left(U(X)\right) \geq \mathbb{E}\left(U(X_0)\right) \text{ for all } U \in \mathcal{U}\right\}$$

We obtain the following theorem:

Theorem 3.1 (Computation of utility price). *The utility price of a strategy leading to the positive contingent claim X_0 from the initial portfolio x_0 satisfies:*

1. *There exists a contingent claim $\tilde{X} \in \mathcal{P}(X_0)$ attainable from the initial portfolio of value $P^{\mathcal{U}}(X_0, x_0)x_0$, i.e.:*

   $$P^{\mathcal{U}}(X_0, x_0) = \min_{X \in \mathcal{P}(X_0)} \pi(X, x_0)$$

   Moreover, $\tilde{X}$ is in the closed convex hull of random vectors distributed as X_0.

2. *The utility price can be also computed with each random vector of pricing $Y \in \mathcal{Y}$ with the formula: $P^{\mathcal{U}}(X_0, x_0) = \sup\left(P(X_0, Y) \mid Y \in \mathcal{D}^{\perp}(x_0)\right)$ and*

$$\begin{aligned} P(X_0, Y) &= \min_{X \in \mathcal{P}(X_0)} \mathbb{E}(YX) \\ &= \mathbb{E}(Y\tilde{X}) \text{ with } \tilde{X} \text{ cyclically anticomonotonic with } Y \text{ and distributed as } X_0 \end{aligned}$$

Before proving this theorem, let us comment some of the results obtained and give some applications. The first one is the existence of an efficient strategy leading to a contingent claim $\tilde{X}$ and giving at least the same expected utility as the contingent claim X_0 for all utility functions $U \in \mathcal{U}$. Nevertheless, this contingent claim is not necessarily distributed as X_0, as it is the case in a complete and frictionless market (see [4]). This implies in particular that for some utility functions $U \in \mathcal{U}$, expected utility of $\tilde{X}$ is strictly bigger than expected utility of X_0. Moreover, another problem which can occur is the fact that supremum in item (2) may be not attained.
However, one can prove that the supremum is a maximum when $\inf ess_{\omega\in\Omega} l(X_0) = 0$. Indeed, in this case the eventual loss of mass for the vector of pricing Y is not important:

Lemma 5 *Let X_0 a random vector in L^1_+ and $x_0 \in int(K)$ such that $\inf essl(X_0) = 0$. For each vector $Y \in \mathcal{D}^\perp(x_0)$, we denote X^aY the random vector distributed as X_0 and cyclically anticomonotonic with Y. Then, the family $(X^aY)_{Y\in\mathcal{D}^\perp(x_0)}$ of random variables is uniformly integrable.*

Proof of the lemma 5.
Let $\varepsilon > 0$. Since $\inf ess_{\omega\in\omega} l(X) = 0$, and each composant of X is positive, we can find ω_0 such that $X^i(\omega_0) \leq \frac{\varepsilon}{2}$ for each $i \in \{0, ..., d\}$. Defining $A^c_\varepsilon = \{X \npreceq \varepsilon \mathbf{1}_0\}$, we prove:
There exists $m \in \mathbb{R}$, such that each $Y \in \mathcal{D}^\perp(x_0)$ is bounded by m on A^c_ε. Indeed, let $A_{\frac{\varepsilon}{2}} = \left\{X \preceq \frac{\varepsilon}{2}\mathbf{1}_0\right\}$. By hypothesis, $p_{\frac{\varepsilon}{2}} \triangleq \mathbb{P}\left(A_{\frac{\varepsilon}{2}}\right) > 0$. Since the portfolio $l(x_0)\mathbf{1}$ is evidently hedged by a strategy from the initial holdings vector x_0, we have $\mathbb{E}(Y_0) < \frac{1}{l(x_0)}$, for each $Y \in \mathcal{D}^\perp(x_0)$. By Tchebychev, we deduce the existence of $m > 0$ such that for all $i \in \{0, ..., d\}$, $\inf_{\omega\in A_{\frac{\varepsilon}{2}}} Y_i \leq m$.
Furthermore, if $\omega_1 \in A^c_\varepsilon$, and $\omega_2 \in A_{\frac{\varepsilon}{2}}$, we have $X(\omega_1) - X(\omega_2) \succeq \frac{\varepsilon}{2}\mathbf{1}_0$. By anticomonotonicity $Y(\omega_1) - Y(\omega_2) \notin K^*$. Since $\frac{Y_i}{Y_j} \leq \frac{\min_{i\in\{0,\dots,d\}} \lambda_i}{\max_{i\in\{0,\dots,d\}} \lambda_i} \triangleq \beta$, we deduce $\beta Y(\omega_2) - Y(\omega_1) \in K^*$. In consequence, $Y_i(\omega) \leq \beta m$ for $i \in \{0, ..., d\}$ and $\omega \in A^c_\varepsilon$.
conclusion: We have:

$$\begin{aligned}\int_{\{X^aY\geq r\}} X^aY d\mathbb{P} &\leq \int_{\{X^aY\geq r\}\cap A_\varepsilon} X^aY d\mathbb{P} + \int_{\{X^aY\geq r\}\cap A^c_\varepsilon} X^aY d\mathbb{P} \\ &\leq \frac{\varepsilon}{2}\int_{\{X^aY\geq r\}\cap A_\varepsilon} \sum_{i=0}^d Y_i d\mathbb{P} + \beta m \int_{\{X^aY\geq r\}\cup A^c_\varepsilon} \sum_{i=0}^d X_i d\mathbb{P}\end{aligned}$$

But since for each $Y \in \mathcal{D}^\perp(x_0)$, $\sum_{i=1}^d Y_i \leq v_{x_0}(\mathbf{1})$, and X_i is integrable, we can choose $r > 0$ such that:

$$\int_{\{X^aY\geq r\}} X^aY d\mathbb{P} \leq \frac{\varepsilon}{2}(1 + v_{x_0}((1))$$

which prove the uniform integrability of the family $(X^aY)_{Y\in\mathcal{D}^\perp(x_0)}$. $\square$.

This uniform integrability helps us to prove the following corollary of theorem (3.1).

Corollary 3.2 *Let $X_0 \in \mathcal{X}^1$, a positive contingent claim with* $\inf ess_{\omega\in\Omega} l(X_0) = 0$*. The supremum in item (2) of theorem (3.1) is a maximum.*

Proof of the corollary 3.2.
Indeed, Let Y_n^a cyclically anticomonotonic with X_0 and distributed as $Y_n \in \mathcal{D}^{\perp}(x_0)$ such that

$$\lim_{n\to+\infty} \mathbb{E}\left(XY_n^a\right) = P^{\mathcal{U}}\left(X_0, x_0\right)$$

From the lemma of Komlòs (and the convexity of $\mathcal{D}^{\perp}(x_0)$), we could suppose that Y_n^a converges to a random vector Y^a distributed as $Y \in \mathcal{D}^{\perp}(x_0)$ and cyclically anticomonotonic with X_0 . Moreover, from the uniform integrability of the sequence $Y_n^a X_0$, we conclude:

$$\mathbb{E}\left(Y^a X\right) \triangleq \lim_{n\to+\infty} \mathbb{E}\left(XY_n^a\right) = P^{\mathcal{U}}\left(X_0, x_0\right)$$

$\square$.

3.1 Demonstration of the computation of inefficiency size

We split the demonstration of the theorem in several lemmata. We have the same result as Jouini and al.([10]) which is a characterization of stochastic dominance of order 2:

Lemma 6 *The set $\mathcal{P}(X_0)$ is closed with respect to the topology of convergence in measure. We have more precisely the following result:*

$$\mathcal{P}(X_0) = \Sigma_0(X_0) + L^1\left(K, \mathcal{F}_T\right)$$

where $\Sigma_0(X_0)$ is the closed convex hull of the contingent claims X distributed as X_0.

Proof of the lemma 6.
Step 1 - $\mathcal{P}(X_0)$ is closed w.r. to the topology of convergence in measure
Indeed, let $X_n \in \mathcal{P}(X_0)$ which converges almost surely to a random variable X^*. By the theorem of dominated convergence, we have if an utility function U^b is bounded:

$$\mathbb{E}(U^b(X^*)) = \lim_{n\to+\infty} \mathbb{E}(U^b(X_n) \geq \mathbb{E}(U^b(X_0)$$

But, now, if U is not bounded, we can construct a sequence U_n^b to U, such that, for all n and $x \in \mathbb{R}^*$, we have $|U_n^b(x)| \leq |U(x)|$. Therefore, with the dominated convergence theorem:

$$\mathbb{E}(U(X^*)) = \lim_{n\to+\infty} \mathbb{E}(U_n^b(X) \geq \mathbb{E}(U(X_0) = \lim_{n\to+\infty} \mathbb{E}(U_n^b(X_0)$$

Step 2 - Characterization of $\mathcal{P}(X_0)$.
Since $\mathcal{P}(X_0)$ is closed for the topology of convergence in measure, we deduce $\Sigma_0(X_0) + L^1(K, \mathcal{F}_T) \subset \mathcal{P}(X_0)$.

Reciprocally, let $X^* \notin \Sigma_0(X_0) + L^1(K, \mathcal{F}_T)$. Since $\Sigma_0(X_0) + L^1(K, \mathcal{F}_T)$ is closed on the space $L^1(\mathcal{R}^d, \mathcal{F}_T)$, we can find an $Y_0 \in L^\infty(\mathcal{R}^d, \mathcal{F}_T)$, such that:

$$\mathbb{E}\left(X^* Y_0\right) < \inf\left\{\mathbb{E}(XY_0) \mid \text{ for all } X \in \Sigma(X_0) + L^1(\mathbb{R}^d, \mathcal{F}_T)\right\}$$

This implies in particular that $Y \in K^*$ a.s. Moreover, it is possible to choose a random vector $\tilde{X}_0$ distributed as X_0, cyclically anticomonotonic with Y_0. There exists therefore a concave function U (see [14]), not necessarily strictly increasing, with $dom(U) \subset K$ and:

$$Y \in \partial U(X) \text{a.s.}$$

We deduce that

$$\mathbb{E}\left(U(X^*)\right) - \mathbb{E}\left(U(\tilde{X}_0)\right) \leq \mathbb{E}\left(Y_0(X^* - \tilde{X}_0)\right) < 0$$

Unfortunately, U may be not in $\mathcal{U}$. Nevertheless, we can choose a sequence of utility functions $U_n \in \mathcal{U}$ (i.e. in particular strictly increasing) such that $U_{n+1}(x) \leq U_n(x)$, $\mathbb{E}\left(U_1(\tilde{X}_0)\right) < \infty$ and $\mathbb{E}\left(U_1(X^*)\right) < \infty$. With the theorem of convergence monotone, we have $\lim_{n\to+\infty} \mathbb{E}(U_n(\tilde{X}_0)) = \mathbb{E}(U(\tilde{X}_0))$, and $\lim_{n\to+\infty} \mathbb{E}(U_n(X^*)) = \mathbb{E}(U(X^*))$. And we deduce the existence of $\tilde{U} \in \mathcal{U}$, with:

$$\mathbb{E}\left(\tilde{U}(X^*)\right) < \mathbb{E}\left(\tilde{U}(X_0)\right) = \mathbb{E}\left(\tilde{U}(X_0)\right)$$

X^* doesn't belong to $\mathcal{P}(X_0)$. This proves that $\mathcal{P}(X_0) = \Sigma_0(X_0) + L^1(K, F_T)$ $\square$.

As it is suggested in the last item of theorem (3.1), computation of utility price is done in two steps. First, we look for the contingent claim $\tilde{X}_Y \in \mathcal{P}(X_0)$ which minimizes $\mathbb{E}(XY)$ for each vector of pricing $Y \in \mathcal{D}^\perp(x_0)$. In a second step, we use duality results to prove that $P^{\mathcal{U}}(X_0, x_0)$ is then the supremum of these computations.
The following lemma is the first step of our demonstration. However, we need to work with only a part of the set $\mathcal{D}^\perp(x_0)$:

Lemma 7 *Let $Y \in \mathcal{D}^\perp(x_0)$ with $\mathbb{P}(Y = 0) = 0$. There exists $\tilde{X}$, distributed as X_0, cyclically anticomonotonic with Y, such that:*

$$\sup_{U \in \mathcal{U}} \inf_{X \in B^U(X_0)} \mathbb{E}(YX) = \min_{X \in \mathcal{P}(X_0)} \mathbb{E}(YX) = \mathbb{E}(Y\tilde{X})$$

Proof of the lemma 7.
Let $X_n \in \mathcal{P}(X_0)$ such that $\lim_{n\to+\infty} \mathbb{E}(X_n Y) = \inf_{X \in \mathcal{P}(X_0)} \mathbb{E}(XY)$. From the lemma of Komlòs, and the lemma of Fatou, we could suppose, with maybe an extraction that X_n converges a.s. to $\tilde{X} \in \mathcal{P}(X_0)$ and:

$$\mathbb{E}\left(\tilde{X}Y\right) = \min_{X \in \mathcal{P}(X_0)} \mathbb{E}(XY)$$

Moreover, from the lemma 6, we have necessarily $\tilde{X} \in \Sigma(X_0)$. Then, the demonstration of propostion 2.2 tells us that $\tilde{X}$ is distributed as X_0 and is cyclically

anticomonotonic with X_0.
The next step is to prove that $\sup_{U\in\mathcal{U}} \inf_{X\in B^U(X_0)} \mathbb{E}(XY) = \min_{X\in\mathcal{P}(X_0)} \mathbb{E}[XY] = \lambda_{opt}$. We have evidently $\sup_{U\in\mathcal{U}} \inf_{X\in B^U(X_0)} \mathbb{E}(XY) \leq \min_{X\in\mathcal{P}(X_0)} \mathbb{E}[XY]$. For the converse inequality, suppose first that X_0 and Y verify also the last item of theorem (2.2). This proves that $\tilde{X}$ is solution to the problem for a certain utility function $U \in \mathcal{U}$:

$$\sup_{X\in\mathcal{X}(\lambda_{opt}x_0)} \mathbb{E}(U(X))$$

and this implies in particular that $\sup_{U\in\mathcal{U}} \inf_{X\in B^U(X_0)} \mathbb{E}(XY) \geq \lambda_{opt}$. However, it is possible that the last item of theorem (2.2) is not satisfied. Let $\varepsilon > 0$. We can choose a random vector ε_X cyclically anticomonotonic with Y, such that $\sup essl(\varepsilon_X) = +\infty$ and $\mathbb{E}(Y\varepsilon_X) \leq \varepsilon$. Let's define the random vector $\tilde{X}$ with

$$\begin{aligned}
\tilde{X} &= X \text{ for } \omega \in \Omega_1 = \{X \preceq x_1\} \\
\tilde{X} &= x_1 \text{ for } \omega \in \Omega_i \\
\tilde{X} &= x_2 + \varepsilon_X \text{ for } \omega \in \Omega_2
\end{aligned}$$

in such a way that, for all $\lambda \in K^*$, $\mathbb{E}\left(\langle\lambda, \tilde{X} - X\rangle\right) = 0$. With this definition, it is easy to see that the random vector $\tilde{X}$ is cyclically anticomonotonic with Y. We first deduce that:

$$\sup_{U\in\mathcal{U}} \inf_{X\in B^U(\tilde{X})} \mathbb{E}(XY) = \mathbb{E}\left(Y\tilde{X}\right)$$

Moreover, X is a portfolio preferred by each agent to $\tilde{X}$. To see this, let $U \in \mathcal{U}$ and $Y \in \partial U(X)$. We have:

$$\mathbb{E}\left(U(\tilde{X})\right) - \mathbb{E}(U(X)) \leq \mathbb{E}\left(\langle Y, \tilde{X} - X\rangle\right)$$

Now if we choose $\lambda \in K^*$ such that $\sup ess\, l(Y)_{|\Omega_1} < l^*(\lambda) < l_2 = \inf ess\, l(Y)_{|\omega_2}$. We have then, a.s., $\langle\lambda, \tilde{X} - X\rangle \geq \langle Y, \tilde{X} - X\rangle$, and we deduce:

$$\mathbb{E}\left(U(\tilde{X})\right) - \mathbb{E}(U(X)) \leq \lambda\mathbb{E}\left(\tilde{X} - X\right) = 0$$

$\square$.

The next element to prove the theorem is to know if it is indeed possible to work by random vector of pricing. The next lemma is a duality result which answers this question.

Lemma 8 *Let $X_0 \in \mathcal{X}^1$ and $x_0 \in K$. We have*

$$\begin{aligned}
\inf_{X\in\mathcal{P}(X_0)} \pi(X, x) &= \min_{X\in\mathcal{P}(X_0)} \sup_{Y\in\mathcal{D}^\perp(x_0)} \mathbb{E}(YX) \\
&= \sup_{Y\in\mathcal{D}^\perp(x_0)} \min_{X\in\mathcal{P}(X_0)} \mathbb{E}(YX)
\end{aligned}$$

Proof of the lemma 8.
Let $n > 0$ and $\mathcal{D}_n^{\perp}(x_0) = \mathcal{D}^{\perp}(x_0) \cap \{Y \mid 0 \leq Y \leq n\}$. The set $\mathcal{D}_n^{\perp}(x_0)$ is a weak convex compact subset of L^1. Moreover with the theorem of dominated convergence, $Y \mapsto \mathbb{E}(YX)$ is continuous. By a theorem minimax (see, for example, theorem 45.8 p. 239, in [15]), we obtain:

$$\sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \inf_{X\in\mathcal{P}(X_0)} \mathbb{E}(YX_0)) = \inf_{X\in\mathcal{P}(X_0)} \sup_{\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}(YX) \tag{3}$$

Moreover, it is easy to see:

$$\lim_{n\to+\infty} \sup_{\mathcal{D}_n^{\perp}(x_0)} \inf_{X\in\mathcal{P}(X_0)} \mathbb{E}(YX) \leq \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \inf_{X\in\mathcal{P}(X_0)} \mathbb{E}(YX) \tag{4}$$

On the other hand, since $\mathcal{P}(X_0)$ is convex and closed for the topology of convergence in measure, we can find, with the help of KOMLÒS theorem, a sequel $X_n \in \mathcal{P}(X_0)$ such that:

$$\lim_{n\to+\infty} \inf_{X\in\mathcal{P}(X_0)} \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}(YX) = \lim_{n\to+\infty} \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}(YX_n)$$

and $X_n \to \hat{X}$ almost surely, where $\hat{X}$ is in $\mathcal{P}(X_0)$. Moreover, we have:

$$\begin{aligned}\inf_{k\geq n} \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}(YX_k) &\geq \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \inf_{k\geq n} \mathbb{E}(YX_k)\\ &\geq \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}\left(\inf_{k\geq n} YX_k\right)\end{aligned}$$

and, therefore with the theorem of monotone convergence

$$\begin{aligned}\lim_{n\to+\infty} \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}\left(\hat{Y}_T X_n\right) &\geq \lim_{n\to+\infty} \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}\left(Y \inf_{k\geq n} X_k\right)\\ &= \sup_{Y\in\mathcal{D}^{\perp}(x_0)} \mathbb{E}\left(Y\hat{X}\right)\end{aligned}$$

As $\hat{X} \in \mathcal{P}(X_0)$, we conclude that:

$$\inf_{X\in\mathcal{P}(X_0)} \sup_{Y\in\mathcal{D}^{\perp}(x_0)} \mathbb{E}(YX) \leq \sup_{Y\in\mathcal{D}^{\perp}(x_0)} \inf_{X\in\mathcal{P}(X_0)} \mathbb{E}(YX)$$

The other inequality is straightforward and the result is proved.

With these two results we can prove finally theorem (3.1).
Proof of the theorem 3.1.
We have:

$$P^{\mathcal{U}}(X_0, x_0) = \sup_{U\in\mathcal{U}} \inf_{X\in B^U(X_0)} \pi(X, x_0) = \sup_{U\in\mathcal{U}} \sup_{Y\in\mathcal{D}^{\perp}(x_0)} \inf_{X\in B^U(X_0)} \mathbb{E}(YX)$$

therefore:

$$P^{\mathcal{U}}(X_0, x_0) \geq \sup_{Y\in\mathcal{D}^{\perp,*}(x_0)} \sup_{U\in\mathcal{U}} \inf_{X\in B^U(X_0)} \mathbb{E}(YX)$$

where $\mathcal{D}^{\perp,*}(x_0) = \{Y \in \mathcal{D}^{\perp}(x_0) \mid \mathbb{P}(Y = 0) = 0\}$. From lemma 7:

$$\sup_{U \in \mathcal{U}} \inf_{X \in B^U(X_0)} \mathbb{E}(YX) = \min_{X \in \mathcal{P}(X_0)} \mathbb{E}(YX)$$

for $Y \in \mathcal{D}^{\perp}(x_0)$ and therefore:

$$P^{\mathcal{U}}(X_0, x_0) \geq \sup_{Y \in \mathcal{D}^{\perp,*}(x_0)} \min_{X \in \mathcal{P}(X_0)} \mathbb{E}\left(YX\right)$$

Moreover, let $Y^* \in \mathcal{D}^{\perp,*}(x_0)$, $Y \in \mathcal{D}^{\perp}(x_0)$, and $Y_\lambda = \lambda Y + (1-\lambda)Y^*$. We have $Y_\lambda \in \mathcal{D}^{\perp,*}(x_0)$ and for all $X \in \mathcal{P}(X_0)$, $\lim_{\lambda \to 1} \mathbb{E}\left(Y_\lambda X\right) = \mathbb{E}(YX)$. We deduce that:

$$\sup_{Y \in \mathcal{D}^{\perp}(x_0)} \min_{X \in \mathcal{P}(X_0)} \mathbb{E}\left(YX\right) = \sup_{Y \in \mathcal{D}^{\perp,*}(x_0)} \min_{X \in \mathcal{P}(X_0)} \mathbb{E}\left(XY\right)$$

Now, we can use lemma (8), and we obtain:

$$P^{\mathcal{U}}(X_0, x_0) \geq \min_{X \in \mathcal{P}(X_0)} \sup_{Y \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\left(YX\right) \tag{5}$$

The other inequality is straightforward and We conclude that:

$$\begin{aligned} P^{\mathcal{U}}(X_0, x_0) &= -\min_{X \in \mathcal{P}(X_0)} \sup_{Y \in \mathcal{D}^{\perp}(x_0)} \mathbb{E}\left(YX\right) \\ &= \inf_{X \in \mathcal{X}} \{\pi(X, x_0) \mid \mathbb{E}\left(U(X)\right) \geq \mathbb{E}\left(U(X_0)\right)\} \\ &= \min_{X \in \mathcal{P}(X_0)} \{\pi(X, x_0) \mid X \text{ is a convex comb. of bundles distributed as } X_0\} \end{aligned}$$

i.e. we have item (1) and (2). But we deduce equally from (5) that :

$$P^{\mathcal{U}}(X_0, x_0) = \sup_{Y \in \mathcal{D}^{\perp}(x_0)} \min_{X \in \mathcal{P}(X_0)} \mathbb{E}\left(YX\right)$$

which gives again with lemma (8) item (3) and concludes the proof. □.

3.2 Characterization of efficient contingent claims

The notion of inefficiency size allows us to define a new notion of efficiency:

Definition 3.2 *Let an admissible trading strategy L with initial value x_0 which leads to the positive contingent claim X_0. X_0 is efficient with respect to the initial portfolio x_0 if his inefficiency cost is zero.*

With this definition, we see evidently that a strictly efficient contingent claim is efficient but the converse is not true in general. To describe correctly the set of efficient contingent claims, we would like to have a similar theorem of characterization as the one for the case of strictly efficient contingent claims. First note that this set is stable with the addition of a cash endowment.

Lemma 9 *Let X_0 an efficient contingent claim with respect to the initial portfolio x_0, and l_0 a cash endowment. Then the contingent claim $X_0 + l_0$ is efficient for the initial portfolio $x_0 + l_0$.*

Proof of the lemma 9.
Indeed, suppose that $X_0 + l_0$ is not efficient. From the theorem (3.1), there exists a strategy which leads to a contingent claim $\tilde{X}$ belonging to the closed convex set of random vectors distributed as $X_0 + l_0$ from an initial portfolio $\lambda(x_0 + l_0)$, with $\lambda < 1$. Let $\tilde{X}_1 = \tilde{X} - l_0$. $\tilde{X}$ belongs to the set $\Sigma_0(X_0)$, and is attainable from an initial portfolio λx_0, and implies in particular that X_0 is not efficient with respect to the initial portfolio x_0.

We prove the following theorem which characterizes efficient contingent claims:

Theorem 3.3 *A positive contingent claim $X_0 \in \mathcal{X}^1$ with $l_0 \triangleq \inf essl(X_0)$ is efficient if and only if there exists an initial portfolio x_0, and $y_0 \in K^*$, $Y_0 \in \mathcal{Y}(y_0)$ such that:*

- $\mathbb{E}\left(Y_0(X_0 - l_0)\right) = y_0(x_0 - l_0)$
- *the random vectors X_0 and Y_0 are cyclically anticomonotonic.*

Proof of the Theorem 3.3.
First, we note that, since the cone of efficient contingent claim is stable with the addition of a cash endowment, we can restrict ourself to the case where $\inf ess\, l(X_0) = 0$.
First implication. Suppose the existence of Y_0 with the properties required by the theorem. With item (3) of theorem (3.1), we deduce the utility price of X_0 with respect to x_0 is 1: X_0 is efficient.
Converse implication. Again, with theorem (3.1) an corollary (3.2) , if X_0 is efficient, there exists an Y_0 cyclically anticomonotone with X_0 such that

$$P^U(X_0, x_0) = 1$$

i.e. there exists $y_0 \in K^*_{x_0}$ such that $Y_0 \in \mathcal{Y}(y_0)$ and:

$$\mathbb{E}\left(X_0 Y_0\right) = x_0 y_0 = 1$$

$\square$.

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